



2. Just before midday the body of an apparent homicide victim is found in a room that is kept at a constant temperature of  $70^{\circ}\text{F}$ . At 12 noon the temperature of the body is  $82^{\circ}\text{F}$  and at 1 pm it is  $76^{\circ}\text{F}$ . Assume that the temperature at the time of death was  $98.6^{\circ}\text{F}$  and that it has cooled in accordance with Newton's law of cooling (the diffusion model). What was the time of death?

3. Suppose that a falling hailstone with mass  $m = \frac{4}{3}\pi r^3$  starts from rest with negligible radius  $r = 0$ . Thereafter, its radius is  $r = kt$  ( $k$  is a constant) as it grows by accretion during its fall.

- (a) Use Newton's second law—according to which the net force  $F$  acting on a possibly variable mass  $m$  equals the time rate of change  $\frac{dp}{dt}$  of its momentum  $p = mv$ —to set up and solve the initial value problem

$$\frac{d}{dt}(mv) = mg, \quad v(0) = 0, \quad (1)$$

where  $m$  is the variable mass of the hailstone,  $v = \frac{dy}{dt}$  is its velocity, and the positive  $y$ -axis points downward.

- (b) Show that the hailstone falls as though it were under  $\frac{1}{4}$  the influence of gravity.

#### 4. Salmon Smolt Migration Model

Salmon spend their early life in rivers, and then swim out to sea where they live their adult lives and gain most of their body mass. When they have matured, they return to the rivers to spawn. Usually they return with uncanny precision to the river where they were born, and even to the very spawning ground of their birth.

The salmon run is the time when adult salmon, which have migrated from the ocean, swim to the upper reaches of rivers where they spawn on gravel beds. Unfortunately, the building of dams and the reservoirs produced by these dams have disrupted both the salmon run and the subsequent migration of their offspring to the ocean.

Luckily, the problem of how to allow the adult salmon to migrate upstream past the tall dams has been solved with the introduction of fish ladders and in a few circumstances fish elevators. These devices allow the salmon to rise up in elevation to the level of the reservoir and thus overcome the dam.

However, the reservoirs still cause problems for the new generation of salmon. About 90 to 150 days after deposition, the eggs or roe hatch. These young salmon called fry remain near their birthplace for 12 to 18 months before traveling downstream towards the ocean. Once they begin this migration to the ocean they are called smolts.

The problem is that the reservoirs tend to be quite large and the smolt population literally becomes far less concentrated in the reservoir water than their original concentration in the stream or river which fed the reservoir. Thus the water exiting the reservoir through the spillway has a very low concentration of smolts. This increases the time required for the migration. The more time the smolts spend in the reservoir, the more likely it is that they will be preyed upon by larger fish.

The question is how to speed up smolt migration through reservoirs in order to keep the salmon population at normal levels.

- (a) Let  $s(t)$  be the number of smolts in the reservoir. It is impractical to measure the concentration of smolts in the river which feeds the reservoir (the tank). Instead we will assume that the smolts arrive at a steady rate,  $r$  which has units of fish/day. If we assume the smolts spread out thoroughly through the reservoir, then the outflow concentration of the smolts is simply the number of smolts in the reservoir,  $s(t)$  divided by the volume of the reservoir which for this part of the problem we will assume remains constant,  $v$ . Finally, assume the outflow of water from the reservoir is given by  $f$ , then we get the following IVP:

$$s' = r - \frac{s}{v}f \quad s(0) = s_0$$

Use the integrating factor method to show that the solution to this IVP is:

$$s(t) = \frac{vr}{f} + Ce^{-\frac{ft}{v}} \quad C = s_0 - \frac{vr}{f}.$$



- (b) Assume the following values, and keep all water measurements in terms of millions of gallons.

$$r = 1000 \text{ fish/day}$$

$$v = 50 \text{ million gallons}$$

$$f = 1 \text{ million gallons/day}$$

$$s_0 = 25000 \text{ fish}$$

How many fish are initially exiting the reservoir per day?

- (c) How many days will it take for the smolt population in the reservoir to reach 40000?

- (d) One way to allow the smolts to pass through the reservoir more quickly is to draw down the reservoir. This means letting more water flow out than is flowing in. Reducing the volume of the reservoir increases the concentration of smolts resulting in a higher rate of smolts exiting the reservoir through the spillway.

This situation can be modeled by the following IVP:

$$s' = r - \frac{s}{v + \Delta f t} \cdot f_{\text{out}} \quad s(0) = s_0,$$

where  $\Delta f = f_{\text{in}} - f_{\text{out}}$ . Use this model and

$$f_{\text{in}} = 1 \text{ mil gal/day}$$

$$f_{\text{out}} = 2 \text{ mil gal/day}$$

to find a function  $s(t)$  which gives the number of smolts,  $s$ , in the reservoir at time  $t$ .



- (e) How many days will it take to reduce the smolt population from 25000 down to 20000? And what will the volume of the reservoir be?