§ 9.8 Taylor and Maclaurin Series

We've looked at ways of manipulating power series to get new power series. Now we ask the obvious question:

Q: Can any function be represented by a power series?
A: (forthcoming)

Let's assume we can represent any function by a power series centered on \(a\), i.e., \(f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots\)

What do higher derivatives of \(f(x)\) say about the \(a_n\)'s?

\[f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \ldots\]
\[f''(x) = 2a_2 + 3 \cdot 2 a_3(x-a) + 4 \cdot 3 a_4(x-a)^2 + 5 \cdot 4 a_5(x-a)^3 + \ldots\]
\[f'''(x) = 3 \cdot 2 a_3 + 4 \cdot 3 \cdot 2 a_4(x-a) + 5 \cdot 4 \cdot 3 a_5(x-a)^2 + 6 \cdot 5 \cdot 4 a_6(x-a)^3 + \ldots\]

Let \(x = a\) and solve for \(a_n\) in each equation above.

\[a_n = \frac{f^{(n)}(a)}{n!}\]

The above analysis tells us two things:
1. The coefficients \(a_n\) depend on \(f\).
2. The coefficients are unique for each \(f\).
**Def** A **Maclaurin series** is a power series representation of a function centered around 0.

**Def** A **Taylor Series** is a power series representation of a function centered around a constant "a".

**Ex** Taylor Series

\[ f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \]

Maclaurin Series

\[ f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots \]

So a Maclaurin Series is just a special case of a Taylor series with the constant \( a = 0 \).

**Ex** What is the Maclaurin series of \( f(x) = 3x^2 + 2x + 1 \) ?

Recall: \( c_n = \frac{f^{(n)}(a)}{n!} \) here \( a = 0 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(0) )</th>
<th>( c_n = \frac{f^{(n)}(0)}{n!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3x^2 + 2x + 1</td>
<td>1</td>
<td>( \frac{1}{0!} = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>6x + 2</td>
<td>2</td>
<td>( \frac{2}{1!} = 2 )</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>0</td>
<td>( \frac{0}{2!} = 0 )</td>
</tr>
</tbody>
</table>

\[ \Rightarrow f(x) = 1 + 2x + 3x^2 + 0x^3 + 0x^4 + \cdots \]

\[ = 1 + 2x + 3x^2 \]

\( \Rightarrow \) so the idea agrees with polynomial functions.
Ex. What is the Taylor Series of \( f(x) = 3x^2 + 2x + 1 \) centered at \( x = 2 \)?

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(2) )</th>
<th>( C_n = \frac{f^{(n)}(2)}{n!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 3x^2 + 2x + 1 )</td>
<td>( 3 \cdot 2^2 + 2 \cdot 2 + 1 = 17 )</td>
<td>( \frac{17}{0!} = 17 )</td>
</tr>
<tr>
<td>1</td>
<td>( 6x + 2 )</td>
<td>( 6 \cdot 2 + 2 = 14 )</td>
<td>( \frac{14}{1!} = 14 )</td>
</tr>
<tr>
<td>2</td>
<td>( 6 )</td>
<td>( 6 )</td>
<td>( \frac{6}{2!} = 3 )</td>
</tr>
<tr>
<td>3</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

\[ f(x) = 17 + 14(x-2) + 3(x-2)^2 + O(x-2)^3 + \cdots \]

Is this really the same function? Let's expand...

\[ f(x) = 17 + 14x - 28 + 3(x^2 - 4x + 4) \]
\[ = 17 - 28 + 14x + 3x^2 - 12x + 12 \]
\[ = 3x^2 + 2x + 1 \checkmark \text{ They do match!} \]

**Thm A. Uniqueness Theorem**

If \( f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \) has radius of convergence \( R > 0 \),

Then \( c_n = \frac{f^{(n)}(a)}{n!} \).

**Remark** Having non-zero radius of convergence implies \( f(x) \) is differentiable and we can differentiate \( f(x) \) indefinitely many times.
Ex. Find the Maclaurin series for \( f(x) = e^x \).

Solution. Maclaurin means \( a = 0 \) in \( f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \).

\[
\begin{array}{c|c|c|c}
 n & f^{(n)}(x) & f^{(n)}(0) & \frac{f^{(n)}(0)}{n!} = a_n \\
\hline
 0 & e^x & e^0 = 1 & 1 \\
 1 & e^x & e^0 = 1 & \frac{1}{1!} = 1 \\
 2 & e^x & e^0 = 1 & \frac{1}{2!} \\
 3 & e^x & e^0 = 1 & \frac{1}{3!} \\
 \vdots & \vdots & \vdots & \vdots \\
 n & e^x & e^0 = 1 & \frac{1}{n!} \\
\end{array}
\]

\[
\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots
\]

What is the radius of convergence, \( R \)? Use Absolute Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \right| \cdot \left| \frac{n!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1} \cdot x}{x^n} \right| \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{|x|}{n+1} = 0
\]

So \( \rho = 0 \) for all real \( x \) \( \Rightarrow R = \frac{1}{\rho} = \frac{1}{0} = \infty \)

\( \Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges absolutely for all real \( x \)!
Ex. Find the Taylor Series for \( f(x) = \sin x \) centered on \( a = 2\pi \).

**Solution**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(2\pi) )</th>
<th>( \frac{f^{(n)}(2\pi)}{n!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sin(x) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \cos(x) )</td>
<td>1</td>
<td>( \frac{1}{1!} )</td>
</tr>
<tr>
<td>2</td>
<td>( -\sin x )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( -\cos x )</td>
<td>-1</td>
<td>( \frac{-1}{3!} )</td>
</tr>
<tr>
<td>4</td>
<td>( \sin x )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \Rightarrow \sin(x) = (x - 2\pi) - \frac{1}{3!} (x - 2\pi)^3 + \frac{1}{5!} (x - 2\pi)^5 - \frac{1}{7!} (x - 2\pi)^7 + \ldots \]

\[ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x - 2\pi)^{2n+1} \]

Radius of convergence, \( R \)?

**Remark:** The radius of convergence will be the same regardless of whether we expand around \( a = 2\pi \) or \( a = 0 \) (Maclauren), thus to make calculations easier set \( a = 0 \).

\[ \lim_{n \to \infty} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \left| \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n \to \infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} |x^2| = 0 \]

\( 2(n+1)+1 = 2n+3 \quad \rho = 0 \Rightarrow R = \frac{1}{\rho} = \infty \)

Note: \( 2(n+1)+1 = 2n+3 \quad \rho = 0 \Rightarrow R = \frac{1}{\rho} = \infty \)

Check the next page to see graphs of \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad N=1, \ldots, 13 \)
The above plot shows the graph of $\sin(x)$ and graphs of partial sums of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, that is, it plots $f(x) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for $N = 1, 3, 5, 7, 9, 11, 13$. 

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Ex. We can differentiate the Taylor series for \( \sin(x) \) centered on \( 2\pi \) to get a Taylor series for \( \cos(x) \) @ \( 2\pi \):

\[
\frac{d}{dx} \left( \sin(x) = (x - 2\pi) - \frac{1}{3!} (x - 2\pi)^3 + \frac{1}{5!} (x - 2\pi)^5 - \frac{1}{7!} (x - 2\pi)^7 + \cdots \right)
\]

\[
\cos(x) = 1 - \frac{3}{3!} (x - 2\pi)^2 + \frac{5}{5!} (x - 2\pi)^4 - \frac{7}{7!} (x - 2\pi)^6 + \cdots
\]

\[
\cos(x) = 1 - \frac{1}{2!} (x - 2\pi)^2 + \frac{1}{4!} (x - 2\pi)^4 - \frac{1}{6!} (x - 2\pi)^6 + \cdots
\]

Radius of convergence? Recall that differentiating or integrating a power series term by term does not change the radius of convergence, i.e. the new series has the same radius of convergence as the previous one.

\( R \) for \( \sin(x) \) was \( \infty \) \( \Rightarrow \) \( R \) for \( \cos(x) \) is \( \infty \) too!

Note: We could have expanded \( \sin(x) \) around any integral multiple of \( \pi \), say \( n\pi \), then if \( n \) is even we get the same exact results as above. If \( n \) is odd, then this is equivalent to multiplying every term by \((-1)^n\). So usually we just give the power expansion of trig functions around 0:

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots
\]

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots
\]
The Taylor polynomial of order $n$ based at $a$ is
\[
P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\]
\[P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k\]

The $n$th Taylor remainder is
\[R_n(x) = f(x) - P_n(x)\]

**Thm.** A function equals its Taylor series at $x$ if
\[
\lim_{n \to \infty} R_n(x) = 0.
\]

**Note:** Your text defines $R_n(x)$ slightly differently so as to make the proof of the above theorem easier. The definition I gave above is more natural and intuitive.

Now we can finally answer the question at the beginning:

Q: Can any function be represented by a power series?

A: No, only if
\[
\lim_{n \to \infty} R_n(x) = 0
\]

Ex. If \( f(x) = \begin{cases} e^{-\frac{1}{2}x^2} & x \neq 0 \\ 0 & x = 0 \end{cases} \) then \( R_n(x) \to 0 \) as \( n \to \infty \).

However, many functions can be faithfully represented.