§ 9.6 Power Series

We are now ready to make the conceptual leap from infinite series of real numbers to infinite series of functions of a real variable.

Ex. Fourier Series

A Fourier Series is simply an infinite series where the functions of $x$ being summed are sines and cosines:

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin(nx)$$

When you graph the above Fourier series you get a square wave:

\[ \begin{array}{cccccc}
    & & & & & \\
    & & & & & \\
    & & & & & \\
\end{array} \]

We will not study Fourier series in this course. We will study a simpler case called power series, where the functions of $x$ are just positive integral powers of $x$, or $(x-a)$.

example

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

\[ \text{note we usually start at zero now.} \]
Def A power series in $x$ is a function of $x$.

The natural question to ask when confronted with a given power series is: "For what values of $x$ does the power series converge?"

Ex. What if $a_n = a$ for all $n$ (i.e., all the coefficients in the power series are the same constant)?

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + ax^4 + \ldots$$

Notice that if we reindex by setting $n = k-1$, we get the geometric series:

$$\sum_{n=0}^{\infty} ax^n \quad \rightarrow \quad \sum_{k=1}^{\infty} ax^{k-1} = \sum_{k=1}^{\infty} a (\frac{x}{1-x})^{k-1} = \frac{a}{1-x}$$

which only holds for $-1 < x < 1$, i.e., $|x| < 1$, so the power series $\sum_{n=0}^{\infty} ax^n$ converges on the set $[-1 < x < 1]$.

Thm Convergence set of a power series

The convergence set for a power series $\sum_{n=0}^{\infty} a_n x^n$ is always an interval of one of the three types:

1) The single point $x = 0$.
2) An interval $(-R, R)$, plus possibly one or both endpoints.
3) The whole real line, $(-\infty, \infty)$.

The three cases are said to have radius of convergence, $0, R, \infty$. 
Ex. What is the convergence set for \( \sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n} \)?

Solution: Use the Absolute Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)2^{n+1}} \cdot \frac{(n+1)2^n}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{2} = \frac{|x|}{2} = \rho
\]

Recall that we get convergence when \( \rho < 1 \), thus

\[
\frac{|x|}{2} < 1 \iff |x| < 2 \iff \boxed{-2 < x < 2}.
\]

Also recall that the test is inconclusive when \( \rho = 1 \), i.e.

When \( x = 2 \) or \( x = -2 \), so we examine these cases separately:

\[
\begin{align*}
  x = 2 & \quad \sum_{n=0}^{\infty} \frac{2^n}{(n+1)2^n} = 1 + 1 + \frac{1}{3} + \frac{1}{4} + \cdots \text{ the harmonic series (which diverges)} \\
  x = -2 & \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n+1)2^n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ the alternating harmonic series (which converges conditionally)}
\end{align*}
\]

\[
\Rightarrow \quad \boxed{-2 \leq x < 2} \text{ is the convergence set}.
\]

Ex. Find the radius of convergence of \( \sum_{n=0}^{\infty} \frac{n! x^n}{3^n} \).

Solution: Use the absolute ratio test:

\[
\lim_{n \to \infty} \left| \frac{(n+1)! x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n! x^n} \right| = \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{3^n}{3 \cdot 3^n} \cdot \left| \frac{x}{3} \right| = \frac{|x|}{3} = \rho
\]

Thus when \( \rho < 1 \) the power series will converge \( \Rightarrow |x| < 3 \).

Test the end-points \( x = 3 \) and \( x = -3 \), plug in and apply any test.

\[
\begin{align*}
  x = 3 & \quad \sum_{n=0}^{\infty} \frac{n! x^n}{3^n} = \sum_{n=0}^{\infty} \frac{n! 3^n}{3^n} = \sum_{n=0}^{\infty} n! \text{ diverges} \Rightarrow \text{don't include } x = 3,
\end{align*}
\]

\[
  x = -3 & \quad \sum_{n=0}^{\infty} \frac{n! x^n}{3^n} = \sum_{n=0}^{\infty} \frac{n! (-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n n! 3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n n! \text{ converges by alternating series test} \Rightarrow \text{include } x = -3.
\]
Goal for next three sections:

Find power series expansions of functions.

We know \( \frac{a}{1-x} = \sum_{n=0}^{\infty} ax^n \) \( \Rightarrow \) \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \)

Our first and most important function!

1) Replace "x" in \( \frac{1}{1-x} \) with "\( x^2 \)"
\[
\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots
\]

2) Replace "x" in \( \frac{1}{1-x} \) with "\( -x^2 \)"
\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots
\]

3) Replace "x" in \( \frac{1}{1-x} \) with "\( x-1 \)"
\[
\frac{1}{2-x} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots
\]

Ex Find a power series representing: \( \frac{x}{1-x^2} \)
\[
= x \cdot 1 + x \cdot (x^2)^1 + x \cdot (x^2)^2 + x \cdot (x^2)^3 + \ldots
\]
\[
= x + x^3 + x^5 + x^7 + \ldots
\]
\[
= \sum_{n=0}^{\infty} x^{(2n+1)}
\]