§ 9.3 Positive Series: The Integral Test

**Thm. Bounded Sum Test**

A series \( \sum a_k \) of nonnegative terms converges \( \iff \) its accompanying sequence of partial sums is bounded above.

**Example**

Show that the series \( \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \) converges.

**Solution**

Note that \( n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \) "n factorial"

Also note that:

\[
\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \cdots
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2
\]

The bottom series is the geometric series with \( a=1, r=\frac{1}{2} \) and thus sums to \( \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2 \). Therefore, by the bounded sum test, given above, the series \( \sum_{n=1}^{\infty} \frac{1}{n!} \) converges.

**Thm. Integral Test** (very important test)

Let \( f \) be a continuous, positive, nonincreasing function on \([1, \infty)\) and suppose \( a_k = f(k) \) for all positive integers \( k \), then

\[
\sum_{k=1}^{\infty} a_k \text{ converges } \iff \int_1^{\infty} f(x) \, dx \text{ converges.}
\]

Def (see your textbook)
Ex: Use the integral test to determine whether the series: \( \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{11}} + \cdots \) converges or diverges.

**Solution**

In "\( \Sigma \)" notation we have \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \).

So \( a_n = \frac{1}{\sqrt{2n+1}} \Rightarrow f(x) = \frac{1}{\sqrt{2x+1}} \) which is positive and non-increasing on \([1, \infty)\) thus the integral test applies:

\[
\int_{1}^{\infty} \frac{1}{\sqrt{2x+1}} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{\sqrt{2x+1}} \, dx = \lim_{b \to \infty} \left[ \sqrt{2x+1} \right]_{1}^{b} = \lim_{b \to \infty} \left( \sqrt{2b+1} - \sqrt{3} \right) = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \text{ diverges.}
\]

**Def** A series of the form \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) where \( p \) is a constant is called a \( p \)-series.

**Thm** \( p \)-series test

1) \( p > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges.} \)

2) \( p \leq 1 \Rightarrow \text{ diverges.} \)

**Proof** Use the integral test with \( f(x) = \frac{1}{x^p} \). Note we solved this integral carefully in lecture notes for section 8.4, and the \( p < 0 \) case diverges by the \( n \)-th term test (section 9.2).
Ex. Does \( \sum_{k=1}^{\infty} \frac{1}{k^{1.0001}} \) converge or diverge?

Solution: This is a p-series with \( p = 1.0001 > 1 \) thus it converges by the p-series test.

Note: The question of convergence or divergence of a series is determined completely by the tail of the series. By this we mean that you may safely ignore any finite number of terms at the beginning of a series without affecting the convergence or divergence of the series as a whole.

Ignoring terms will change the sum if the series converges, but this is usually easy to compensate for.