7.2 Integration by Parts

Recall the product rule, where \( u = u(x) \), \( v = v(x) \):

\[
D_x [u \cdot v] = u'v + uv' \quad \text{product rule}
\]

\[
D_x [u \cdot v] = vdu + udv
\]

\[
\int D_x [u \cdot v] dx = \int vdu + \int udv
\]

\[
u \cdot v = \int vdu + \int udv
\]

\[
\Rightarrow \int udv = uv - \int vdu
\]  \quad \text{memorize this!}

Notice there is a \( dv \) on the left and only \( v's \) on the right hand side of the equation.

This means: If we can break up an integrand into two factors, \( u \) and \( dv \), and we can integrate \( dv \) to get \( v \), then we can rewrite the difficult integral on the left using the "simpler" rewrite rule on the right.

Similar to how u-substitution "undoes" the chain rule of differentiation, one can think of integration by parts as "undoing" the product rule of differentiation, however this is only an analogy.
Ex. Find $\int x \cos x \, dx$

$u = x \quad dv = \cos x \, dx$
$du = dx \quad v = \sin x$

When doing integration by parts, I recommend writing both substitutions, i.e. $u \& dv$ in the top quadrants of a box.

\[
\int \frac{x \cos x \, dx}{u \ dv} = \frac{x \sin x}{1} - \int \frac{\sin x \, dx}{v} - \int v \, du
\]
\[
= x \sin x - (\cos x) + C
\]
\[
= x \sin x + \cos x + C
\]

Ex. $\int \frac{\ln x \, dx}{u \ dv}$

$u = \ln x \quad dv = dx$
$du = \frac{1}{x} dx \quad v = x$

We can finally integrate $\ln x$!

\[
\int \ln x \, dx = (\ln x) \cdot x - \int x \cdot \frac{1}{x} \, dx
\]
\[
= x \ln x - \int dx
\]
\[
= x \ln x - x + C
\]

Ex. $\int \arcsin x \, dx$

$u = \arcsin x \quad dv = dx$
$du = \frac{1}{\sqrt{1-x^2}} \, dx \quad v = x$

Now $u$-substitute!

\[
\frac{\arcsin x \, dx}{u \ dv} = x \cdot \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx
\]
\[
= x \cdot \arcsin x - \frac{1}{2} \int u^{1/2} \, du
\]
\[
= x \cdot \arcsin x + \frac{1}{2} \cdot 2 \cdot u^{1/2} + C
\]
\[
= x \cdot \arcsin x + \frac{1}{2} \cdot \sqrt{1-x^2} + C
\]
How to choose \( u \) & \( dv \):

**LIPET**

Guideline for how to choose \( u \), whatever is leftover becomes \( dv \).

Memorize This!

**Ex.** \( \int 2x e^{3x} \, dx \)

\[
\int 2x e^{3x} \, dx = \frac{2}{3} x e^{3x} - \frac{2}{9} e^{3x} + C
\]

**Ex.** \( \int e^{x} \cos x \, dx \)

\[
\int e^{x} \cos x \, dx = e^{x} \sin x - \int e^{x} \sin x \, dx
\]

\[
= e^{x} \sin x - \left[ -e^{x} \cos x + \int e^{x} \cos x \, dx \right]
\]

\[
\Rightarrow \int e^{x} \cos x \, dx = \frac{1}{2} e^{x} \left[ \sin x + \cos x \right] + C
\]
Tabular Method: A fast way to integrate \((poly) \cdot \{\text{exp, trig}\}\).

To find \(\int (P(x) \cdot E(x)) \, dx\), write 3 columns:

- **Polynomial**
- **Trigonometric or exponential function** (e.g., \(\sin x, e^{2x}\))

<table>
<thead>
<tr>
<th>du derivatives</th>
<th>(\int dv) integrals</th>
<th>sign(+)</th>
<th>-</th>
<th>+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consecutive derivatives of (P(x))</td>
<td>Consecutive integrals of (E(x))</td>
<td>go here.</td>
<td>go here.</td>
<td>-</td>
</tr>
</tbody>
</table>

First row is always:

| \(P(x)\) | \(E(x)\) | - |

**Ex.** \(\int (2x^2 - 3) e^x \, dx = (2x^2 - 3)e^x - 4xe^x + 4e^x + C\)

<table>
<thead>
<tr>
<th>(du)</th>
<th>(\int dv)</th>
<th>(+/-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2x^2 - 3)</td>
<td>(e^x)</td>
<td>-</td>
</tr>
<tr>
<td>(4x)</td>
<td>(e^x)</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>(e^x)</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>(e^x)</td>
<td>-</td>
</tr>
</tbody>
</table>
\[ \int x^3 \sin \left(\frac{x}{2}\right) \, dx = -2x^3 \cos \left(\frac{x}{2}\right) + 12x^2 \sin \left(\frac{x}{2}\right) + 48x \cos \left(\frac{x}{2}\right) - 96 \sin \left(\frac{x}{2}\right) + C \]

<table>
<thead>
<tr>
<th>(du)</th>
<th>(\int dV)</th>
<th>(+/−)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^3)</td>
<td>(\sin \left(\frac{x}{2}\right))</td>
<td>−</td>
</tr>
<tr>
<td>(3x^2)</td>
<td>−2 (\cos \left(\frac{x}{2}\right))</td>
<td>+</td>
</tr>
<tr>
<td>(6x)</td>
<td>−4 (\sin \left(\frac{x}{2}\right))</td>
<td>−</td>
</tr>
<tr>
<td>6</td>
<td>8 (\cos \left(\frac{x}{2}\right))</td>
<td>+</td>
</tr>
<tr>
<td>0</td>
<td>16 (\sin \left(\frac{x}{2}\right))</td>
<td>−</td>
</tr>
</tbody>
</table>

You can always check your work in this column by starting at the last entry and differentiating until you hopefully get the first entry.

Hint: This is a great type of test question!
\[ \int_{0}^{1} \frac{1}{t+1} \, dt = (t-1)^{12}, \quad \frac{t^2}{2} \bigg|_{0}^{1} - \int_{0}^{1} \frac{t^2}{2} \, (t-1)^{11} \, dt \]

\[ u = (t-1)^{12}, \quad \frac{dv}{dt} = t \, dt \]
\[ du = 12(t-1)^{11} \, dt, \quad v = \frac{t^2}{2} \]

\[ \int_{0}^{1} \frac{1}{t+1} \, dt = \frac{1}{13} (t-1)^{13} \bigg|_{0}^{1} - \int_{0}^{1} \frac{1}{13} (t-1)^{13} \, dt \]

\[ = -\frac{1}{13} \cdot \frac{1}{14} (t-1)^{14} \bigg|_{0}^{1} \]
\[ = \frac{-1}{13 \cdot 14} (1 - 0)^{14} \]
\[ = \frac{-1}{182} \]

Try again!

The above demonstrates:
\[ \int_{a}^{b} u \cdot dv = uv \bigg|_{a}^{b} - \int_{a}^{b} v \cdot du \]

that is, the method of integration of parts works for definite integrals too!