§ 6.5 Exponential Growth and Decay

Population Model

Suppose we want to be able to predict the world population in the year 2020. That is we want a model of the population, a function that allows us to put in a time t and get out $P(t)$, i.e. the # of people living on the planet, at time t.

Assumptions

1. The change in population ΔP (over a period of time Δt) should be proportional to number of people.

$$\Delta P = kP$$

k ← proportionality constant.

Essentially we're saying that larger populations have more babies than smaller ones and vice versa.

2. The change in population ΔP should be proportional to the amount of time Δt between measurements, i.e.

$$\Delta P = kP\Delta t$$

Essentially we're saying that given more time a population tends to have more babies and vice versa.

Let's rearrange:

$$\frac{\Delta P}{\Delta t} = kP,$$ now we'll take the limit as $t \to 0$,

$$\boxed{\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = kP}$$

(Ignore the fact that populations change by discrete integer amounts.)

We can solve this 1st order differential equation by separating variables and integrating:

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt$$

$$\Rightarrow \int \frac{dP}{P} = \int k\,dt$$

$$\Rightarrow \ln|P| = kt + C$$

$$\Rightarrow e^{\ln|P|} = e^{kt+C} = e^{kt}e^{C}$$

$$\Rightarrow |P| = e^{kt}\cdot e^{C}$$

Since negative population doesn't make sense, and because the right-hand side is always >0, let's drop the absolute value on P.

$$\Rightarrow P(t) = e^{kt}\cdot e^{C}$$

Suppose that initially, i.e. at $t=0$, $P(t=0) = 10{,}000$

Then $P(0) = 10{,}000 = e^{k\cdot 0}\cdot e^{C}$ Recall $e^{0} = 1$, thus

$$P(0) = 10{,}000 = 1\cdot e^{C} \Rightarrow e^{C} = 10{,}000.$$

We will use a shorthand notation $P_0 = P(0) = 10{,}000$;

$$\boxed{P(t) = P_0 e^{kt}}$$

Q: How do we figure out a value for k (the growth rate)?

A: We need to know the population at some other time.

$\longrightarrow$

Suppose after 10 years, i.e. $P(10) = 12{,}000$, then what is k?

$$P(10) = 12{,}000 = 10{,}000\, e^{k \cdot 10}$$

$$\Rightarrow \quad \frac{12{,}000}{10{,}000} = e^{k \cdot 10}$$ now undo exp with ln.

$$\Rightarrow \quad \ln(1.2) = \ln(e^{10k})$$

$$\Rightarrow \quad \ln(1.2) = 10k$$

$$\Rightarrow \quad k = \frac{10}{\ln(1.2)} \approx .01823$$

$$\Rightarrow \quad \boxed{P(t) = P_0 e^{kt} = 10{,}000\, e^{.01823t}}$$

So assuming population growth remains constant, we can model any population just by knowing the actual population at two different points in time.

Doubling Time

Q: When will the population double?

A: Solve $2P_0 = P_0 e^{kt}$ for t_2:

$$\Rightarrow \quad \frac{2\cancel{P_0}}{\cancel{P_0}} = e^{kt_2} \Rightarrow \ln 2 = \ln(e^{kt_2}) = kt_2 \Rightarrow \ln 2 = kt_2$$

$$\Rightarrow \quad t_2 = \frac{\ln 2}{k} = \frac{\ln 2}{.01823} \approx \boxed{38 \text{ years}}$$

Exponential Decay & Half Life

Most of the carbon in the world is ^{12}C (carbon-12). The 12 comes from 6 neutrons and 6 protons. However there are small amounts of ^{14}C in our atmosphere. This atmospheric ^{14}C is produced when high energy cosmic rays (in this case neutrons) collide with nitrogen:

$$^{1}n + {}^{14}N \longrightarrow {}^{14}C + {}^{1}p.$$

The resulting ^{14}C is radioactive and will eventually beta-decay to ^{14}N:

$$^{14}_{6}C \longrightarrow \underset{\text{nitrogen}}{^{14}_{7}N} + \underset{\text{electron}}{e^{-}} + \underset{\text{anti-neutrino}}{\bar{\nu}_e}$$

The <u>half-life</u> of ^{14}C is <u>5730 years</u>. This is the time it takes for half of the ^{14}C in a sample to decay to $^{14}_{7}N$.

$$\tfrac{1}{2}P_0 = P_0 e^{kt} \Rightarrow \tfrac{1}{2}\cancel{P_0} = \cancel{P_0} e^{k\cdot 5730}$$

$$\Rightarrow \ln\left(\tfrac{1}{2}\right) = k\cdot 5730$$

$$\Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{5730} = \frac{\ln 1 - \ln 2}{5730} = \frac{-\ln 2}{5730}$$

$$k \approx -0.000120981$$

negative for exponential decay.

$$\boxed{k \approx -1.20981 \times 10^{-4}}$$

Carbon Dating

All living things contain ^{14}C. While alive, the ratio of ^{14}C isotopes to ^{12}C remains constant. When an organism dies it no longer absorbs (via consumption or respiration) the ^{14}C isotope. Thus if we can determine this ratio, then we can date many ancient remains.

Ex. If charred logs of an old fort show only 70% of the ^{14}C expected in living trees, when did the fort burn down. (Assume the fort was built soon after the logs were cut down. Note that burning does not change nuclear structure, only chemical structure.)

Solution

$$y = y_0 e^{kt}$$

$$0.7 \cancel{y_0} = \cancel{y_0} e^{kt}$$

$$\ln 0.7 = \cancel{\ln}(\cancel{e}^{kt}) = kt$$

$$t = \frac{\ln 0.7}{k} = \frac{\ln 0.7}{-1.20981 \times 10^{4}}$$

$$t \approx 2949$$

$$\boxed{\text{So about 2950 years ago}}$$

* Also make sure you understand Newton's Law of Cooling!

Newton's Law of Cooling

Let $T(t)$ be the temperature T as a function of time t.

$$\frac{dT}{dt} = k(T - T_1)$$

where T_0 = initial temperature of object cooling
T_1 = ambient room temperature.

In words, Newton's Law of Cooling says that:

"The rate of change in temperature is proportional to the difference in temperature between the object and its surrounding medium."

Problem A batch of brownies is taken from a 350°F oven and placed in a refrigerator at 40°F and left to cool. After 15 minutes, the brownies have cooled to 250°F. When will the brownies be at 110°F?

① determine k: $T_1 = 40°F$ $T_0 = 350°F$

$$\int \frac{dT}{T-40} = \int k\,dt \Rightarrow \ln|T-40| = kt + C$$

$$|T-40| = e^{kt+C} = e^{kt}e^{C} = T_0 e^{kt}$$

$$\Rightarrow |T-40| = T_0 e^{kt}$$

Now use the info in the 2nd sentence:

$$250 - 40 = 350 e^{k\cdot(1/4)} \qquad 15 \text{ minutes} = 1/4 \text{ hr.}$$

$$\frac{210}{350} = e^{k(1/4)}$$

$$\ln\left(\frac{210}{350}\right) = \frac{1}{4}k \Rightarrow k = 4\ln\left(\frac{210}{350}\right) \approx -2.0433$$

$\longrightarrow$

2 Now solve $|110-40| = 350\, e^{-2.0433t}$

for t:

$$70 = 350\, e^{-2.0433t}$$

$$\ln\left(\frac{70}{350}\right) = \cancel{\ln}\left(\cancel{e}^{-2.0433t}\right)$$

$$\ln\left(\frac{70}{350}\right) = -2.0433t$$

$$t = \frac{-1}{2.0433} \ln\left(\frac{70}{350}\right)$$

$$t \approx .79 \text{ hrs} \qquad (.79\text{ hr.})\left(\frac{60\text{ min}}{\text{hr.}}\right) \approx \boxed{47\text{ mins}}$$