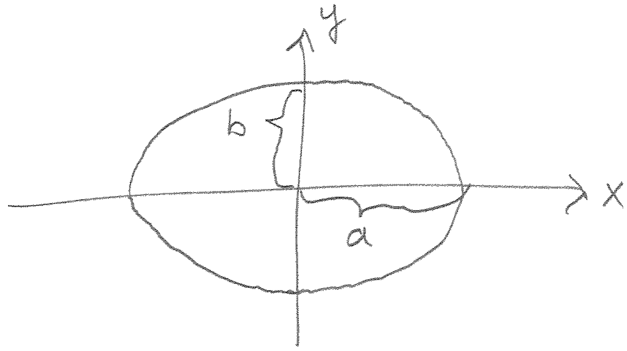
§10.4 Parametric Representation of Curves in the Plane

(First a crash course in ellipses and circles)

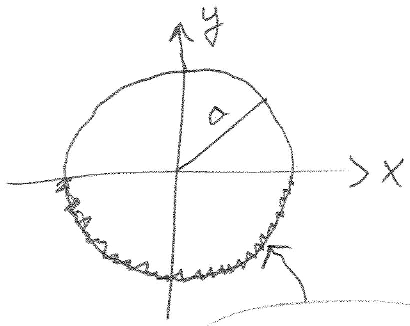
The equation of an <u>ellipse</u> is given by: $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$



If $a=b$ then you have a <u>circle</u> and we usually write its equation $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \implies \boxed{x^2 + y^2 = a^2}$

where a = radius.

Solve for y!



$$y^2 = a^2 - x^2$$

$$\boxed{y = \pm\sqrt{a^2 - x^2}}$$

not a function (it's multivalued)

$y = -\sqrt{a^2 - x^2}$ bottom half

(end crash course)

<u>Def</u> A <u>parametric representation of a curve</u> is a mapping of time to the plane (or 3-space, 4-space, etc.)

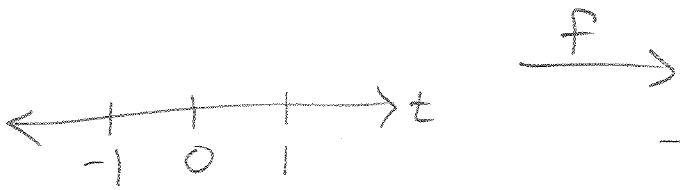
$$f : \mathbb{R} \to \mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$$

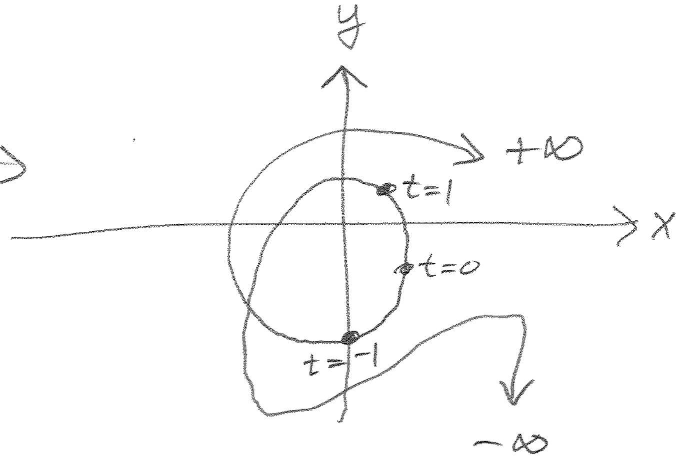
time $\to$ 2-space (the plane)

<u>the parameter</u>

$$f(t) = (x(t), y(t))$$

You should visualize parametrically defined curves as an embedding of the real number line into the x-y plane a.k.a. 2-space ($\mathbb{R}^2$).





Think of it as lifting the real line above up out of the paper (or screen) and stretching, twisting it and laying it back down.

Perhaps an even more fruitful way to think of it is to imagine that the curve represents the position of a small charged particle say an electron as it moves due to an electric field (which might also be changing with time).

<u>Ex</u> Parametric representation of an ellipse:

$$\boxed{f(t) = (\underbrace{a\cos(t)}_{x(t)}, \underbrace{b\sin(t)}_{y(t)})}$$

Does this satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

check: $\frac{(a\cos(t))^2}{a^2} + \frac{(b\sin(t))^2}{b^2} \stackrel{?}{=} 1$

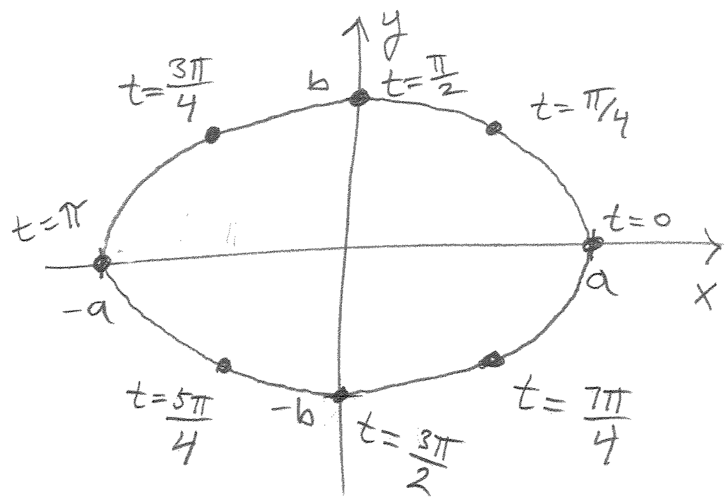
$$\underbrace{\frac{\cancel{a^2}\cos^2(t)}{\cancel{a^2}} + \frac{\cancel{b^2}\sin^2(t)}{\cancel{b^2}}}_{\text{This is the Pythagorean Identity.}} = 1 \checkmark$$

The only way to graph this is to create a table of values

t	x(t)	y(t)
-2	⋮	⋮
-1		
0		
1		
2		

but when we have trig functions for x(t) and or y(t) it makes more sense to use multiples of π. →

t	$a\cos(t)$	$b\sin(t)$
0	a	0
$\pi/4$	$a\,\frac{\sqrt{2}}{2}$	$b\,\frac{\sqrt{2}}{2}$
$\pi/2$	0	b
$3\pi/4$	$-a\,\frac{\sqrt{2}}{2}$	$b\,\frac{\sqrt{2}}{2}$
π	$-a$	0
$5\pi/4$	$-a\,\frac{\sqrt{2}}{2}$	$-b\,\frac{\sqrt{2}}{2}$
$3\pi/2$	0	$-b$
$7\pi/4$	$a\frac{\sqrt{2}}{2}$	$-b\,\frac{\sqrt{2}}{2}$
2π	a	0



How do we do Calculus with parametric curves?

Derivatives: Suppose we want to find the slope of the tangent line to the curve:

Suppose $\boxed{(x, y) = (f(t), g(t))}$ and $f'(t) \neq 0$ (for some interval of time)

Then $\boxed{\begin{array}{l} f : t \to x \\ f^{-1} : x \to t \end{array}}$ $\therefore$ where $\boxed{F = g \circ f^{-1}}$

$$y = g(t) = g(f^{-1}(x)) = F(x) = F(f(t)) \Rightarrow y(t) = F(f(t))$$

$$\Rightarrow \frac{dy}{dt} = F'(f(t)) \cdot f'(t) \quad \text{(by the chain rule)}$$

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Let's use our new formula: $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}}$ to see if we get the same answer as ordinary differentiation gives us for the unit circle:

parametric curve (new)

$$(x, y) = (\overset{x}{\cos t}, \overset{y}{\sin t})$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\cos t}{-\sin t}$$

$$= \frac{x}{-y} = \boxed{\frac{-x}{y}}$$

function $y(x)$ (old)

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \sqrt{1-x^2} \quad \text{(just use top half)}$$

$$y = (1-x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(1-x^2)^{-1/2} \cdot (-\cancel{2}x)$$

$$= \frac{-x}{\sqrt{1-x^2}} = \boxed{\frac{-x}{y}}$$

match!

2nd Derivative

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{d}{dt}(\overset{\frac{dy}{dx}}{y'})}{\frac{dx}{dt}}$$

Ex Let $x = 5\cos t$ $y = 4\sin t$ $0 < t < 3$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-5\sin t} = -\frac{4}{5}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{d}{dt}\left(-\frac{4}{5}\cot(t)\right)}{-5\sin t} = \frac{\frac{4}{5}\csc^2(t)}{-5\sin t} = \frac{-4}{25}\csc^3(t)$$

Integration

$$\boxed{\begin{aligned} x &= 2t-1 \\ y &= t^2+2 \end{aligned}}$$

Ex Evaluate $\int_1^3 y\,dx$

Solution Convert everything to "t" including the limits of integration. @ $x=1$, $1=2t-1 \Rightarrow 2t=2 \Rightarrow t=1$

@ $x=3$, $3=2t-1 \Rightarrow 2t=4 \Rightarrow t=2$

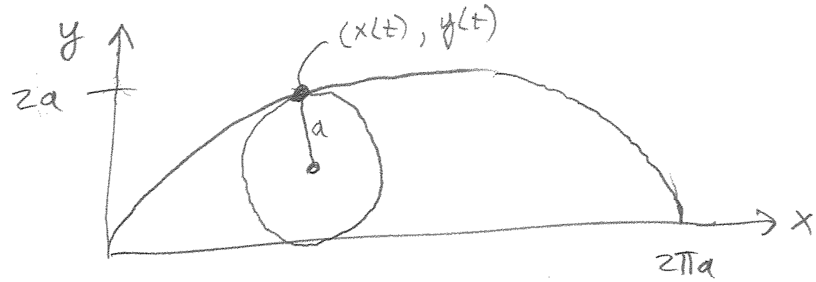
$$\int_1^3 y\,dx = \int_1^2 \underbrace{(t^2+2)}_{y}\,\underbrace{2dt}_{dx} = \int_1^2 (2t^2+4)\,dt = \left(\frac{2}{3}t^3+4t\right)\Big|_1^2$$

$$= \left(\frac{2}{3}\cdot 2^3 + 4\cdot 2\right) - \left(\frac{2}{3}\cdot 1^3 + 4\cdot 1\right) = \left(\frac{16}{3}+8\right) - \left(\frac{2}{3}+4\right)$$

$$= \frac{14}{3}+4 = \frac{14}{3}+\frac{12}{3} = \boxed{\frac{26}{3}}$$

Ex Evaluate

Fleas on a tricycle (see box on p. 534)



$$\boxed{\begin{aligned} x(t) &= a(t - \sin t) \\ y(t) &= a(1 - \cos t) \end{aligned}}$$

$$\frac{dx}{dt} = a - a\cos t = a(1-\cos t)$$

$$\frac{dy}{dt} = a\sin t$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\; dt$$

$$= a\int_0^{2\pi} \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1}}\; dt$$

$$= a\int_0^{2\pi} \sqrt{2(1-\cos t)}\; dt$$

Recall Double-angle formula

$$\cos(2t) = 1 - 2\sin^2(t)$$

$$\cos(t) = 1 - 2\sin^2\left(\tfrac{t}{2}\right)$$

$$1 - \cos(t) = 2\sin^2\left(\tfrac{t}{2}\right)$$

$$= a\int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{t}{2}\right)}\; dt$$

$$= 2a\int_0^{2\pi} \sin\left(\tfrac{t}{2}\right) dt \qquad u = \frac{t}{2} \qquad du = \frac{1}{2}dt \qquad 2\,du = dt$$

$$= 4a\int_{t=0}^{t=2\pi} \sin(u)\, du = -4a\cos\left(\tfrac{t}{2}\right)\Big|_0^{2\pi} = -4a\left[\cos\left(\tfrac{2\pi}{2}\right) - \cos(0)\right]$$

$$= -4a\left[-1-1\right]$$

$$= \boxed{8a}$$