Instructions:

- Answer the questions in the space provided.
- You must show your work in order to get credit! Writing just an answer is worth 0 points, even if the answer is correct.
- Partial credit will be awarded.
- The instructor has extra scratch paper if you need it.
- Graphing and scientific calculators are allowed, but smartphones and computers are not allowed.
- This exam is closed book and closed notes, except you may use one double sided 8.5 by 11 inch page of notes.

1. [10 points] Find $y'$ if $y = \ln \left( \frac{2x}{3x + 4} \right)$.

(Hint: Use the properties of the logarithm to simplify before differentiating.)

\[ y = \ln (2x) - \ln (3x + 4) \]
\[ y' = \frac{1}{2x} \cdot 2 - \frac{1}{3x+4} \cdot 3 \]
\[ y' = \frac{1}{x} - \frac{3}{3x+4} \]
2. [10 points] Let \( y = \frac{\sqrt{x^2 + 1}}{(9x - 4)^2} \). Use logarithmic differentiation to find \( y' \).

\[
\ln y = \ln \left[ \frac{\sqrt{x^2 + 1}}{(9x - 4)^2} \right]
\]

\[
\ln y = \ln (x^2 + 1)^{\frac{1}{2}} - \ln (9x - 4)^2
\]

\[
\ln y = \frac{1}{2} \ln (x^2 + 1) - 2 \ln (9x - 4)
\]

\[
\frac{1}{y} \cdot y' = \frac{1}{2} \cdot \frac{1}{x^2 + 1} \cdot 2x - 2 \cdot \frac{1}{9x - 4} \cdot 9
\]

\[
\frac{1}{y} \cdot y' = \frac{x}{x^2 + 1} - \frac{18}{9x - 4}
\]

\[
y' = \frac{\sqrt{x^2 + 1}}{(9x - 4)^2} \left[ \frac{x}{x^2 + 1} - \frac{18}{9x - 4} \right].
\]
3. [10 points] Find the inverse function of \( f(x) = 2(x-1)^2 - 1 \), \( x > 1 \).

\[
\begin{align*}
  y &= 2(x-1)^2 - 1 \\
  x &= 2(y-1)^2 \\
  x+1 &= 2(y-1)^2 \\
  \frac{x+1}{2} &= (y-1)^2 \\
  \sqrt{\frac{x+1}{2}} &= y-1 \\
  \Rightarrow f^{-1}(x) &= \sqrt{\frac{x+1}{2}} + 1
\end{align*}
\]

Check:
\[
\begin{align*}
  f^{-1}(f(x)) &= f^{-1}\left(2(x-1)^2 - 1\right) \\
  &= \sqrt{\frac{2(x-1)^2 - 1}{2} + 1} \\
  &= \sqrt{(x-1)^2 + 1} = x \\
  f(f^{-1}(x)) &= f\left(\sqrt{\frac{x+1}{2}} + 1\right) \\
  &= 2\left(\sqrt{\frac{x+1}{2}} + 1\right)^2 - 1 \\
  &= 2\left(\frac{x+1}{2} + 1\right)^2 - 1 = 2\cdot\frac{x+1}{2} - 1
\end{align*}
\]

4. [10 points] Graph both the function \( y = 2x + 1 \) and its inverse on the same coordinate axes. (Hint: Also graph the line \( y = x \).)
5. [10 points] Evaluate the expression: \( \ln \left( \frac{1}{\sqrt[e]{e}} \right) = \ln e^{-\frac{1}{3}} = -\frac{1}{3} \ln e^1 = -\frac{1}{3} \)

6. [10 points] Find \( \int \frac{e^{4x}}{1 + e^{4x}} \, dx \).

\[
\int \frac{e^{4x}}{1 + e^{4x}} \, dx = \frac{1}{4} \int \frac{1}{u} \, du
\]

\[
= \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |1 + e^{4x}| + C
\]

7. [10 points] Evaluate \( \int_0^3 2x^2 \, dx \).

\[
u = x^2 \quad du = 2x \, dx \quad \frac{1}{2} \, du = x \, dx
\]

\[
\int_0^3 2x^2 \, dx = \frac{1}{2} \int_0^3 u \, du = \frac{1}{2} \left. \frac{1}{2} u^2 \right|_0^3 = \frac{1}{2} \ln 2 \cdot 2^x \bigg|_0^3
\]

\[
= \left. \frac{1}{2} \ln 2 \left[ 2^9 - 2^0 \right] \right. = \frac{511}{2 \ln 2}
\]
8. [10 points] Solve the population equation below, that is, find \( P(t) \). Show all steps of the solution.

\[
\frac{dP}{dt} = kP
\]

\[\begin{align*}
\text{5} & \quad \frac{dP}{P} = k \, dt \\
\text{6} & \quad \int \frac{dP}{P} = \int k \, dt \\
\text{7} & \quad \ln |P| = kt + C \\
\text{8} & \quad e^{\ln |P|} = e^{(kt+C)} \\
|P| & = e^{kt} \cdot e^C \\
\text{9} & \quad P(t) = e^{kt} \cdot e^C \\
\text{Let } P_0 = P(0) \text{ then } P(0) = e^0 \cdot e^C = P_0 \\
\therefore e^C = P_0, \text{ so} \\
\text{10} & \quad P(t) = P_0 \, e^{kt}
\end{align*}\]
9. [10 points] A bone is found to contain 30% of the carbon-14 that it contained when it was part of a living organism. How long ago did the organism die? (The half-life of carbon-14 is 5730 years.)

First find \( k \):

\[
\frac{1}{2} P_0 = P_0 e^{k \cdot 5730}
\]

\[
\ln\left(\frac{1}{2}\right) = \ln\left(e^{k \cdot 5730}\right) = k \cdot 5730
\]

\[
\Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.209068 \times 10^{-4}
\]

\[
0.3 P_0 = P_0 e^{kt}
\]

\[
\ln(0.3) = \ln\left(e^{kt}\right) = kt
\]

\[
t = \frac{\ln(0.3)}{k} = \frac{\ln(0.3) \cdot 5730}{\ln\left(\frac{1}{2}\right)}
\]

\[
t \approx 9950 \text{ years}
\]
10. [10 points] Find the general solution of the following differential equation. You may assume $x \neq 0$.

\[
\frac{dy}{dx} + \frac{2}{x} y = x^2
\]

\[
I.F. = e^{-\int \frac{2}{x} \, dx} = e^{-2 \int \frac{1}{x} \, dx} = e^{-2 \ln |x|} = e^{-2 \ln (1x^{-2})} = |x|^{-2}
\]

\[
\begin{align*}
\int \left[ y \cdot x^{-2} \right] ' \, dx &= \int 1 \cdot dx \\
\int [y \cdot x^{-2}] ' \, dx &= \int 1 \cdot dx \\
y \cdot x^{-2} &= x + C
\end{align*}
\]

\[
y(x) = x^3 + Cx^2
\]
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