Symmetries of the Square

The purpose of this lecture is to help you see the "big picture" of solving algebraic equations.

First we number the corners of a square like \[
\begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array}
\]
so we can keep track of how each symmetry moves the square in 3 dimensions. There are 8 symmetries, consisting of 4 rotations and 4 flips:

\[
\begin{array}{c}
\begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array} & \xrightarrow{R_{90}} & \begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array} \\
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array} & \xrightarrow{HF} & \begin{array}{c}
3 & 4 \\
2 & 1 \\
\end{array}
\end{array}
\]
horizontal flip

\[
\begin{array}{c}
\begin{array}{c}
1 & 4 \\
2 & 3 \\
\end{array} & \xrightarrow{V F} & \begin{array}{c}
1 & 2 \\
4 & 3 \\
\end{array}
\end{array}
\]
vertical flip

\[
\begin{array}{c}
\begin{array}{c}
4 & 1 \\
1 & 2 \\
\end{array} & \xrightarrow{D_{1\!F}} & \begin{array}{c}
4 & 1 \\
3 & 2 \\
\end{array}
\end{array}
\]
diagonal flip #1

\[
\begin{array}{c}
\begin{array}{c}
3 & 2 \\
4 & 1 \\
\end{array} & \xrightarrow{D_{2\!F}} & \begin{array}{c}
2 & 3 \\
1 & 4 \\
\end{array}
\end{array}
\]
diagonal flip #2

Next, we see that it is very natural to "compose" these symmetries together into chains of operations e.g.

1) \[
\begin{array}{c}
\begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array} & \xrightarrow{R_{90}} & \begin{array}{c}
1 & 4 \\
2 & 3 \\
\end{array} & \xrightarrow{HF} & \begin{array}{c}
2 & 3 \\
1 & 4 \\
\end{array} & \Rightarrow & R_{90} \ast HF = D_{2\!F}
\end{array}
\]

2) \[
\begin{array}{c}
\begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array} & \xrightarrow{D_{1\!F}} & \begin{array}{c}
4 & 1 \\
2 & 3 \\
\end{array} & \xrightarrow{D_{2\!F}} & \begin{array}{c}
4 & 3 \\
1 & 2 \\
\end{array} & \Rightarrow & D_{1\!F} \ast D_{2\!F} = R_{180}
\end{array}
\]
The next natural step is to construct a composition table, similar to a multiplication table except we’re calling the operation which composes two symmetries “*”:

<table>
<thead>
<tr>
<th></th>
<th>$R_0$</th>
<th>$R_{90}$</th>
<th>$R_{180}$</th>
<th>$R_{270}$</th>
<th>HF</th>
<th>VF</th>
<th>$D_1F$</th>
<th>$D_2F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0$</td>
<td>$R_0$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
<td>$R_{270}$</td>
<td>HF</td>
<td>VF</td>
<td>$D_1F$</td>
<td>$D_2F$</td>
</tr>
<tr>
<td>$R_{90}$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
<td>$R_{270}$</td>
<td>$R_0$</td>
<td>$D_2F$</td>
<td>$D_1F$</td>
<td>HF</td>
<td>VF</td>
</tr>
<tr>
<td>$R_{180}$</td>
<td>$R_{180}$</td>
<td>$R_{270}$</td>
<td>$R_0$</td>
<td>$R_{90}$</td>
<td>VF</td>
<td>HF</td>
<td>$D_2F$</td>
<td>$D_1F$</td>
</tr>
<tr>
<td>$R_{270}$</td>
<td>$R_{270}$</td>
<td>$R_0$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
<td>$D_1F$</td>
<td>$D_2F$</td>
<td>VF</td>
<td>HF</td>
</tr>
<tr>
<td>HF</td>
<td>HF</td>
<td>$D_1F$</td>
<td>VF</td>
<td>$D_2F$</td>
<td>$R_0$</td>
<td>$R_{180}$</td>
<td>$R_{90}$</td>
<td>$R_{270}$</td>
</tr>
<tr>
<td>VF</td>
<td>VF</td>
<td>$D_2F$</td>
<td>HF</td>
<td>$D_1F$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
<td>$R_{270}$</td>
<td>$R_{90}$</td>
</tr>
<tr>
<td>$D_1F$</td>
<td>$D_1F$</td>
<td>$VF$</td>
<td>$D_2F$</td>
<td>HF</td>
<td>$R_{270}$</td>
<td>$R_{90}$</td>
<td>$R_0$</td>
<td>$R_{180}$</td>
</tr>
<tr>
<td>$D_2F$</td>
<td>$D_2F$</td>
<td>$HF$</td>
<td>$D_1F$</td>
<td>VF</td>
<td>$R_{90}$</td>
<td>$R_{270}$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
</tr>
</tbody>
</table>

Things to notice about this table:

1) The topmost row is exactly the same as the column headers. Similarly, the leftmost column is exactly the same as the row headers because $R_0 * X = X$ and $X * R_0 = X$.

   We call $R_0$ the identity element.

2) Every element occurs exactly once in each row, and exactly once in each column.
3) Because $R_0$ occurs in every row and every column, if you give me any element, say "x" in the table, you can find another element, say "y" such that $x \ast y = R_0$

e.g. $R_{90} \ast ? = R_0$ answer: $R_{90} \ast R_{270} = R_0$

OR $? \ast HF = R_0$ answer: $HF \ast HF = R_0$

we call these elements inverses and write:

$$R^{-1}_{90} = R_{270} \quad \text{also} \quad R^{-1}_{270} = R_{90}.$$  

4) The operator "\ast" is associative, meaning

$$(x \ast y) \ast z = x \ast (y \ast z)$$

So the way in which you associate two elements together is inconsequential. e.g.

$$(HF \ast R_{90}) \ast D_2 F = HF \ast (R_{90} \ast D_2 F)$$

$$D_1 F \ast D_2 F = HF \ast VF$$

$R_{180} = R_{180}$$
Question: How do you solve an equation like:

\[ R_{270} \times X \times D_2 F = HF \]

Strategy: Isolate the unknown "x" by cleverly using the properties of inverses and the identity element, and the fact that every element has an inverse.

\[
(R_{270}^{-1} \times R_{270}) \times X \times D_2 F = R_{270}^{-1} \times HF
\]

\[
(R_0 \times X) \times D_2 F = R_{270}^{-1} \times HF
\]

\[
X \times D_2 F = R_{270}^{-1} \times HF
\]

\[
X \times (D_2 F \times D_2 F^{-1}) = R_{270}^{-1} \times HF \times D_2 F^{-1}
\]

\[
(X \times R_0) = R_{270}^{-1} \times HF \times D_2 F^{-1}
\]

\[
X = R_{270}^{-1} \times HF \times D_2 F^{-1}
\]

\[
X = R_{90} \times HF \times D_2 F
\]

\[
X = (R_{90} \times HF) \times D_2 F
\]

\[
X = D_2 F \times D_2 F
\]

\[
X = R_0
\]

Check: \[ R_{270} \times R_0 \times D_2 F = R_{270} \times D_2 F = HF \] ✅
A set, $G$ with a binary operation $*: G \times G \rightarrow G$ such that

1) the operation is associative
   \[ (x*y)*z = x*(y*z) \text{ always} \]

2) the set has an identity element "$e"
   \[ e*x = x*e \text{ for all } x \in G \]

3) every element has an inverse
   \[ x^{-1} \in G \text{ such that } x^{-1}*x = e \text{ and } x*x^{-1} = e \]

is called a group.

Example: $(\mathbb{Z}, +)$ where $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

Here the identity $e = 0$.

1) associativity
   \[ (2+3)+(-4) = 2+(3+(-4)) \]

2) identity
   \[ 0+x = x+0 \text{ for any } x \]

3) inverses
   \[ 2+(-2) = 0 = (-2)+2 \]

So the integers form a group under addition.