5.4 Present Value of Annuities

Last time we determined how to compute the future value of an annuity.

$$\boxed{S = R\left(\frac{(1+r_c)^N - 1}{r_c}\right)}$$

S = future value (sum) of annuity

R = regular payment

N = # of payments = nt (t in yrs.)

$r_c = \frac{r}{n} = \frac{\text{interest rate}}{\text{\# compoundings per year}}$

In this scheme we know R, N, r, n and this allows us to determine S, or the value of all our payments plus interest accrued.

Next we looked at a sinking fund, which is a slight variation of the above scheme where instead of knowing R (the regular payment), we know S, the future value of the account, because we will use the account to pay a future debt. This formula is just the above formula solved for R.

$$\boxed{R = S\left(\frac{r_c}{(1+r_c)^N - 1}\right)}$$

Ex. Sinking Fund

Mark has a debt of \$28,000 to pay in 5 years. How much must he invest at the end of each quarter in an account earning 12% interest compounded quarterly to be able to pay off the debt?

Solution:

$S = \$28,000$

$N = 5 \cdot (4) = 20$

$$\left.\begin{array}{l} r = 0.12 \\ n = 4 \end{array}\right\} r_c = \frac{r}{n} = \frac{0.12}{4} = .03$$

$$R = 28,000\left(\frac{0.03}{(1+0.03)^{20}-1}\right) = 28,000\,(0.03721571)$$
$$= \boxed{1,042.04}$$

$20(\$1,042.04) = \$20,840.80 \implies \$28,000$

because of interest!

So Mark saved \$7,159.20 by planning ahead.

Def. present value of an annuity - the lump sum of money that must be in your account now earning compound interest in order to make regular withdrawals later.

In other words, by not withdrawing all of your money now, you allow the decreasing balance to still earn compound interest, which in the long run will get you more money. Here's the formula, which is obtained by summing a geometric sequence similarly to the future value case:

$$\boxed{S = \frac{R\left(1-(1+r_c)^{-N}\right)}{r_c}}$$

Ex. 1 Frank estimates that he'll need to withdraw \$10,000 at the end of every quarter from his retirement account for living expenses for 25 years. The account earns 6.5% interest compounded quarterly. How much money needs to be in the account in order to achieve this goal?

$$S = \frac{10{,}000\left(1-(1.01625)^{-100}\right)}{0.01625}$$

$$\boxed{S = \$492{,}615.03}$$

$R = 10{,}000$

$N = 4 \cdot 25 = 100$

$\left.\begin{array}{l} r = 0.065 \\ n = 4 \end{array}\right\} r_c = \frac{r}{n} = \frac{0.065}{4} = 0.01625$

Def present value of an annuity due – same as the present value of an annuity except withdrawals are made at the beginning of compounding period.

$$\boxed{S_{due} = \frac{R(1+r_c)\left(1-(1+r_c)^{-N}\right)}{r_c} = (1+r_c)S}$$

present value of annuity

Ex. 4 Karen won \$1,200,000 in the state lottery! The prize is awarded by payments of \$5,000 at the beginning of every month for 20 years. If the money is in an account earning 5% interest compounded monthly, what is the value of the prize today?

$R = \$5{,}000 \qquad N = nt = 12(20) = 240$

$r = 0.05 \qquad n = 12 \qquad r_c = \frac{r}{n} = \frac{.05}{12} = .0041\overline{6}$

$$S_{due} = \frac{5{,}000(1+.0041\overline{6})\left(1-(1+.0041\overline{6})^{-240}\right)}{.0041\overline{6}}$$

$$\boxed{= \$760{,}783}$$

Def. Present value of a deferred annuity –

If withdrawals are postponed for m compounding periods, at which point N regular withdrawals of \$$R$ take place.

$$\boxed{P = \frac{R(1-(1+r_c)^{-N})}{r_c(1+r_c)^m}}$$

$m=$ # compounding periods during which no withdrawals are made.

Ex. 6 Upon graduation from college, Luce's parents decide to give her a large cash gift. They give her exactly enough to create monthly retirement payments of \$4,000 paid at the end of each month, starting in 40 years. The account pays 5.4% interest compounded monthly. How much money must they put into the account for Luce upon graduation? (Payments last 20 yrs.)

$R = 4{,}000$ $\quad N = nt = 12 \cdot 20 = 240$ $\quad m = 40 \cdot 12 = 480$

$r = 0.054$ $\quad n = 12$ $\quad r_c = \frac{r}{n} = \frac{.054}{12} = 0.0045$

$$P = \frac{4{,}000(1-(1+0.0045)^{-240})}{(0.0045)(1+0.0045)^{480}}$$

$$= \boxed{\$67{,}942.84}$$