Example: "Compound Interest"

\[ S = P \left(1 + \frac{r}{n}\right)^{nt} \]

- \( S \) = account value
- \( P \) = principal
- \( r \) = interest rate
- \( n \) = \# of compoundings/year
- \( t \) = time (in years)

How long does it take for your initial deposit of \( P \) dollars to double if \( r = 8\% \) and interest is compounded \( 4 \) times annually (\( n=4 \))?

\[ 2P = P \left(1 + \frac{0.08}{4}\right)^{4t} \]

\[ 2 = \left(1 + 0.02\right)^{4t} \]

\[ 2 = (1.02)^{4t} \quad \leftrightarrow \quad \log_{1.02} 2 = 4t \]

\[ \ln 2 = \ln(1.02)^{4t} \]

\[ \ln 2 = 4t \ln(1.02) \]

\[ \frac{\ln 2}{\ln(1.02)} = 4t \]

\[ t = \frac{\ln 2}{4 \ln 1.02} \]

\[ t \approx 8.75 \text{ years} \]
Example 2: "National Debt"

According to data from the Treasury Department, the national debt was about $930 billion in 1980. By 2007, it was approximately $9,008 billion or $9.008 trillion. Assume the debt grows exponentially (with base e), and use the above data to determine the parameters (i.e., the values of $A_0$ and $r$) in the model below.

$$y(x) = A_0 e^{rx}$$

Exponential Growth Model

- $A_0$: initial amount (i.e., amount at $t = 0$)
- $r$: growth rate

Solution:

1. Let $x =$ years since 1980 (1980 is time 0)

$$\Rightarrow y(0) = A_0 e^{0} = A_0 e^0 = A_0 = 930 \text{ billion}$$

So set $A_0 = 930$, we'll count by billions.

2. \[ \frac{2007 - 1980}{27} = \frac{27}{27} = 1 \]

$$\Rightarrow y(27) = 930 e^{1.27} = 9,008$$

$$\Rightarrow e^{1.27} = \frac{9,008}{930}$$

$$\Rightarrow 1.27 = \ln\left(\frac{9,008}{930}\right)$$

$$\Rightarrow r = \frac{1}{27} \cdot \ln\left(\frac{9,008}{930}\right) \approx 0.0841$$
Therefore our model for national debt growth is:

\[ y(x) = 930 e^{0.0841x} \text{ (in billions)} \]

Use this model to estimate:

(a) The year in which the debt will reach $25,000 billion.

(b) The predicted national debt in 2050.

(a) \[ 25,000 = 930 e^{0.0841x} \] solve for \( x \) (years after 1980)

\[ e^{0.0841x} = \frac{25,000}{930} \]

\[ 0.0841x = \ln\left(\frac{2500}{93}\right) \]

\[ x \approx 39.4 \]

\[ x + 1980 = \frac{1980.0 + 39.4}{2019.4} \approx 2019 \]

(b) \[ \frac{2050 - 1980}{70} \Rightarrow y(70) = 930 e^{0.0841 \cdot 70} \]

\[ \approx 335,100 \]

Debt in 2050 \( \approx \frac{335,100}{10^6} \text{ billion} \]

\[ = \frac{335.1}{10^12} \text{ trillion} \]
Example 3 "Richter Scale"

The Richter scale is a logarithmic function which assigns a magnitude to the measured intensity:

\[ M(I) = \frac{2}{3} \log \left( \frac{I}{I_0} \right) \]

where \( I_0 \) is the threshold intensity, or the smallest intensity that can be accurately measured by the seismometer.

(a) If an earthquake measures a magnitude of 4.8, what is the intensity, relative to \( I_0 \)?

\[ 4.8 = \frac{2}{3} \log \left( \frac{I}{I_0} \right) \Rightarrow \left( \frac{3}{2} \right)^{2.4} = \log \left( \frac{I}{I_0} \right) \]

\[ \Rightarrow 7.2 = \log \left( \frac{I}{I_0} \right) \Rightarrow 10^{7.2} = \frac{I}{I_0} \]

\[ \Rightarrow I = 10^{7.2} I_0 \text{ so } I > \frac{10 \text{ million} \cdot I_0}{10^7} = 10^7 \]

\[ I \approx 15,800,000 I_0 \]

(b) If an earthquake is twice as intense as in part (a), what is its magnitude?

\[ M \left( 2 \cdot 10^{7.2} I_0 \right) = \frac{2}{3} \log \left( \frac{2 \cdot 10^{7.2} I_0}{I_0} \right) \]

\[ \approx 5 \]
(c) How much more intensity is measured in a magnitude 7.1 earthquake as compared to a magnitude 4.8 earthquake like in part (a)?

Solution: Let's find the intensity \( I \) of a magnitude 7.1 earthquake and compare it with the intensity found in part (a).

\[
7.1 = \frac{2}{3} \log \left( \frac{I}{I_0} \right) \Rightarrow \frac{3}{2} \cdot 7.1 = \log \left( \frac{I}{I_0} \right)
\]

\[
10^{10.65} = \log \left( \frac{I}{I_0} \right) \Rightarrow 10^{10.65} = \frac{I}{I_0}
\]

\[
\Rightarrow I = 10^{10.65} I_0 \Rightarrow I = 10^{10.65} I_0
\]

\[
\Rightarrow \frac{I_{7.1}}{I_{4.8}} = \frac{10^{10.65} I_0}{10^{7.2} I_0} = 10^{10.65-7.2} = 10^{3.45} \approx 2,818
\]