4.5 Solving Logarithmic and Exponential Equations

A logarithmic equation is an equation with some function of x inside the log function. E.g. $\log_5(7x) - \log_5(x+1) = 1$.

Basic strategy:
1. Use the properties of logarithms to combine all log expressions into one log expression.
2. Rewrite in equivalent exponential form.

   Same as "Apply exponential to both sides of the equation."
3. Use previously learned techniques to solve the resulting equation.
4. Check the domain. Recall that the domain of $\log_a x$ is all positive reals.

Properties of Logarithms

3. $\log_a a^x = x$
4. $a^{\log_a x} = x$ for $x > 0$.
5. $\log_a(m \cdot n) = \log_a m + \log_a n$
6. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
7. $\log_a(m^n) = n \log_a m$

Examples

a) $3\log_2 x = 12 \Rightarrow \log_2 x = \frac{12}{3} = 4$

$\Rightarrow 2^4 = x$

$\Rightarrow x = 16$ Domain: $16 > 0$ ✓

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(b) $\ln(x+3)^8 = 4 \implies 8\ln(x+3) = 4$

$\implies \ln(x+3) = \frac{4}{8} = \frac{1}{2}$

$\implies e^{\frac{1}{2}} = x+3$

$\implies \boxed{x = e^{\frac{1}{2}} - 3}$

Domain ✓ because $(x+3)^8$ always positive!

(c) $\underbrace{\ln(4-x) + \ln 2}_{\text{change a sum of two logs into a log of a product}} = 2\ln x$

$\implies \ln[(4-x)\cdot 2] = \underbrace{2\ln x}_{\text{use exponent rule}}$

$\implies \ln[(4-x)\cdot 2] = \ln x^2$

Now "isolate" all log expressions of the same base to one side of the equation and combine.

$\implies \ln[(4-x)\cdot 2] - \ln x^2 = 0$

$\implies \ln \frac{(4-x)\cdot 2}{x^2} = 0$

$\implies 1 = e^0 = \frac{(4-x)\cdot 2}{x^2} \implies x^2 = 8 - 2x$

$\implies x^2 + 2x - 8 = 0 \implies (x+4)(x-2) = 0$

so $x = -4$ or $x = 2$

Check Domain: RHS of $\ln(4-x) + \ln 2 = 2\ln x$

fails on $x = -4$ because $2\ln(-4)$ is undefined!

$\implies \boxed{x = 2}$

(d) $\log_3 x + \log_3(x-8) = 2$

$\Rightarrow \log_3[x\cdot(x-8)] = 2$

$\Rightarrow 3^2 = x\cdot(x-8) = x^2-8x$

$\Rightarrow x^2-8x-9=0$

$\Rightarrow (x-9)(x+1)=0$

$x=9, \; x=-1$

what is the domain of: $\log_3 x + \log_3(x-8)$?

$x>0$ and $x-8>0$

$x>0$ and $x>8 \Longrightarrow x>8$ is the domain

$\Rightarrow \cancel{x=-1}$ so only $\boxed{x=9}$.

Tricky

(f) $\log_5(x^2) = (\log_5 x)^2$

$2\cdot\log_5(x) = (\log_5(x))^2$

factor out: $\log_5(x)$

$\Rightarrow (\log_5(x))^2 - 2\log_5(x) = 0$

$\Rightarrow \log_5(x)[\log_5(x)-2] = 0 \Rightarrow$ Either: ① $\log_5(x)=0$ OR ② $\log_5(x)-2=0$

① $\log_5(x)=0$

$5^0 = x \Longleftrightarrow \boxed{x=1}$

② $\log_5(x)-2=0 \Rightarrow \log_5(x)=2$

$5^2=x \quad \boxed{x=25}$ Domain ✓

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Solving Exponential Equations

(a) $\left(3^{2x-5}\right)^2 = 3^4$

$3^{(2x-5)2} = 3^4$

$\Rightarrow (2x-5)\cdot 2 = 4$

$4x - 10 = 4$

$4x = 14$

$x = \frac{14}{4} = \boxed{\frac{7}{2}}$

(b) $4(5^x - 1) = 20$

$5^x - 1 = 5$

$5^x = 6 \quad \Rightarrow \quad \log_5(5^x) = \log_5(6)$

$\boxed{x = \log_5(6)}$

(c) $\frac{7}{2^x} = 14$ cross-multiply

$7 = 14 \cdot 2^x \quad \Rightarrow \quad \frac{7}{14} = 2^x \quad \Rightarrow \quad \frac{1}{2} = 2^x$

$\Rightarrow 2^{-1} = 2^x$

$\boxed{x = -1}$

(d) $10^{x+1} + 3(10^x) = 39$

$10^x(10+3) = 39$

$10^x = \frac{39}{13} = 3$

$10^x = 3 \iff \boxed{\log_{10} 3 = x}$

(e) $e^{2x} - 9e^{x} = -20$ This is a quadratic eqn. in disguise.

$(e^x)^2 - 9(e^x) + 20 = 0$

Let $y = e^x$ then:

$y^2 - 9y + 20 = 0$

$\Rightarrow (y-5)(y-4) = 0$

$\Rightarrow (e^x - 5)(e^x - 4) = 0$

So $e^x = 5$ or $e^x = 4$

$\Rightarrow \boxed{x = \ln 5}$ or $\boxed{x = \ln 4}$

Note: There's no need to check the domain of exponential equations because the domain of all exponential equations is $\mathbb{R}$.

Change of Base Formula

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b} = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}}$$

Example

So if you want to calculate

$$\log_2 5 = \frac{\ln 5}{\ln 2}$$

23.) $\ln(x^2-4) + \log_3(x^2-4) = 1$

$$\ln(x^2-4) + \frac{\ln(x^2-4)}{\ln 3} = 1$$

$$(\ln 3)(\ln(x^2-4)) + \ln(x^2-4) = \ln 3$$

$$\ln(x^2-4)(\ln 3 + 1) = \ln 3$$

$$\ln(x^2-4) = \frac{\ln 3}{1+\ln 3}$$

$$x^2-4 = e^{\frac{\ln 3}{1+\ln 3}} =$$

$$x^2 = 4 + e^{\frac{\ln 3}{1+\ln 3}}$$

$$x = \pm\sqrt{4+e^{\frac{\ln 3}{1+\ln 3}}}$$

27.) $\ln(e^x) + e^{\ln(x)} + \ln e = 4$

$$x + x + 1 = 4$$

$$2x + 1 = 4$$

$$2x = 3$$

$$\boxed{x = \frac{3}{2}}$$