4.1 Inverse Functions

Def. If $f(x)$ and $g(x)$ are functions such that

$$f(g(x)) = x = g(f(x))$$

then $g(x)$ is the inverse of $f(x)$ which we denote: $g(x) = f^{-1}(x)$

which is read "f inverse of x."

Example

$$f(x) = 2x + 7 \qquad g(x) = \frac{x-7}{2}$$

$$\left.\begin{aligned} f(g(x)) &= f\left(\frac{x-7}{2}\right) = 2\left(\frac{x-7}{2}\right) + 7 = x \\ g(f(x)) &= g(2x+7) = \frac{(2x+7)-7}{2} = \frac{2x}{2} = x \end{aligned}\right\} g = f^{-1}$$

$$\boxed{\text{Note: } \quad f^{-1}(x) \neq \frac{1}{f(x)}}$$

Q: Are all functions invertible? A: No

Q: When is a function invertible?

Consider $f(x) = x^2$, notice that $f(-5) = f(5) = 25$

so what should $f^{-1}(25)$ be? If $f^{-1}(25) = 5$ and

$f^{-1}(25) = -5$ then f^{-1} is not a function.

Def A function is one-to-one or 1-1 if

$$f(x) = f(y) \Rightarrow x = y.$$

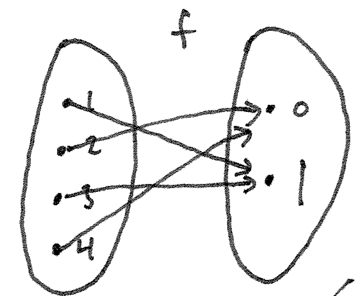
So $f(x) = x^2$ is not 1-1 because $f(-5) = f(5) \not\Rightarrow -5 = 5$.

Q: If $f(x) = x^2$, what should $f^{-1}(-2)$ be?

A: There is no good answer because x^2 is never negative. It doesn't map the real line <u>onto</u> the whole real line (only the positive half).

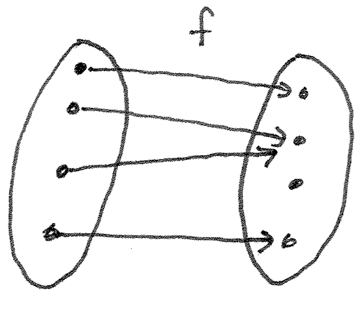
<u>Def</u> A function f is <u>onto</u> if for every y in the range of f, there exists an x in the domain of f such that $f(x) = y$.

<u>Examples</u>



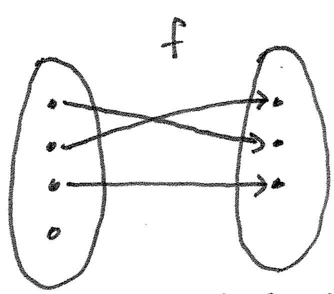
f onto? ✓

f 1-1? X



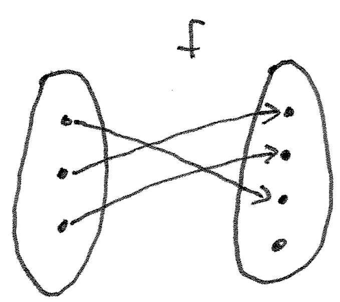
f onto? X

f 1-1? X



f onto? ✓

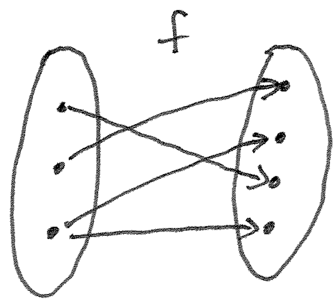
f 1-1? ✓



f onto? X

f 1-1? ✓

Notice that if the range is larger than the domain, then the mapping can't be onto <u>and</u> still be a function.



f onto? ✓

f 1-1? ✓

f a function? X

Fact: A function is invertible, i.e. f^{-1} exists
$\iff$ f is 1-1 and f is onto.
"if and only if"

Finding the Inverse of a function (2 methods)

1) "Pants" technique – Think of a function as an ordered list of operations done to the input, then the inverse function is the list of inverse operations done in reverse order. (Think: getting dressed / undressed, hence "pants".)

Example $f(x) = 4x + 6$

f: 1. multiply input (x) by 4
2. add 6 to previous result

f^{-1}: 1. subtract 6 from input
2. divide previous result by 4.

$$\Rightarrow f^{-1}(x) = \frac{(x-6)}{4}$$

check: $f(f^{-1}(x)) = f\left(\frac{x-6}{4}\right) = 4\left(\frac{x-6}{4}\right) + 6 = (x-6)+6 = x$ ✓

$f^{-1}(f(x)) = f^{-1}(4x+6) = \frac{(4x+6)-6}{4} = \frac{4x}{4} = x$ ✓

2.) Formal Algebraic Technique

a) First swap x and y in the function definition.

b.) Solve for y.

Example

$y = f(x) = 4x + 6$

$x = 4y + 6 \Rightarrow x - 6 = 4y \Rightarrow \frac{x-6}{4} = y$

$\Rightarrow \boxed{f^{-1}(x) = \frac{x-6}{4}}$

Example

$y = f(x) = 2x^3 + 5$

$x = 2y^3 + 5 \Rightarrow x - 5 = 2y^3 \Rightarrow \frac{x-5}{2} = y^3$

$\sqrt[3]{\frac{x-5}{2}} = \sqrt[3]{y^3} \Rightarrow y = \sqrt[3]{\frac{x-5}{2}} \Rightarrow \boxed{f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}}$

check:

$f(f^{-1}(x)) = 2\left(\sqrt[3]{\frac{x-5}{2}}\right)^3 + 5 = 2\left(\frac{x-5}{2}\right) + 5$

$= x - 5 + 5 = x$ ✓

$f^{-1}(f(x)) = \sqrt[3]{\frac{(2x^3+5)-5}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$ ✓