3.6 Rational Functions

**Def.** A rational function is a ratio or quotient of two polynomials.

\[ f(x) = \frac{p(x)}{q(x)} \]

**Ex.**

\[ f(x) = \frac{2x+5}{x-2} \]

In addition to the polynomial graphing "tools" we developed in section 3.5, today we introduce three more:

1. **Vertical asymptotes**
2. **Horizontal asymptote**
3. **Slant asymptote**

These are similar, and each function only has one or none.

1. **Vertical asymptotes** are vertical lines which correspond with the zeros of the denominator polynomial.

**Ex.**

\[ f(x) = \frac{2x+5}{x-2} \Rightarrow \text{vert. asym.} : \boxed{x=2} \]

\[ g(x) = \frac{x}{x^2-9} = \frac{x}{(x+3)(x-3)} \Rightarrow 2 \text{ vert. asym.} \boxed{x=-3} \text{ and } \boxed{x=3} \]

**Note:** A rational function may have several vertical asymptotes, but the function will NEVER cross any of them.

2. **Horizontal asymptote** occurs when \( \deg(p(x)) \leq \deg(q(x)) \)

**Case 1:**

**Equal degrees**

\[ y = \frac{4}{2} = \boxed{y=2} \]

Just take the ratio of the leading coefficients of \( p(x) \) and \( q(x) \).
case 2 \[ \deg(p(x)) < \deg(q(x)) \]

**Ex.** \[ f(x) = \frac{p(x)}{q(x)} = \frac{100x + 50}{x^2 + 1} \Rightarrow H.A. \-y = 0 \]

why? Even though \(100x\) is big compared to \(x^2\) for small values of \(x\), when \(x \to \infty\), the denominator will be much, much bigger than the numerator, and the ratio will essentially be 0.

3.) **Slant asymptote** occurs when \(\deg(p(x)) = \deg(q(x)) + 1\)

**Ex.** \[ f(x) = \frac{2(x+2)(x-2)}{x-3} = \frac{2x^2-8}{x-3} \]

Do polynomial long division:

\[
\begin{array}{c|cc}
 & 2x & +6 \\
\hline 
x-3 & 2x^2 & +0x & -8 \\
 & \text{\underline{\(-(2x^2\ -\ 6x)\)}} &  \\
 & \text{\underline{\(6x\ -\ 8\)}} & \\
 & \text{\underline{\(-(6x\ -\ 18)\)}} & \\
 & & \text{10} \\
\end{array}
\]

The quotient is the equation of the slant asymptote!

**Note:** When \(\deg(p) > \deg(q) + 1\), you still get asymptotes, but we won’t cover that case.
Vertical Asymptotes

The first three lines define a rational function:

> p := x-> 3*x-1: # numerator polynomial
> q := x-> x-4: # denominator polynomial
> f := x-> (3*x-1)/(x-4);

\[ f := x \mapsto \frac{3x - 1}{x - 4} \quad (1.1) \]

Approaching 4 from the left (from negative infinity)

The next command evaluates the polynomial p(x) at the x values given in the array below, notice that as we get closer to 4, the values change very little.

> map(p, [3, 3.5, 3.6, 3.8, 3.9, 3.95, 3.99, 3.999, 3.9999]);
[ 8, 9.5, 9.8, 10.4, 10.7, 10.85, 10.97, 10.997, 10.9997] \quad (1.2)

This next line evaluates the polynomial q(x) at the same x values as above. Notice how the values get very small and approach 0.

> map(q, [3, 3.5, 3.6, 3.8, 3.9, 3.95, 3.99, 3.999, 3.9999]);
[-1, -0.5, -0.4, -0.2, -0.1, -0.05, -0.01, -0.001] \quad (1.3)

Approaching 4 from the right (from positive infinity)

The next command does the same thing as two lines above but from the positive, right hand side of the number line.

> map(p, [5, 4.5, 4.4, 4.2, 4.1, 4.05, 4.01, 4.001, 4.0001]);
[14, 12.5, 12.2, 11.6, 11.3, 11.15, 11.03, 11.003, 11.0003] \quad (1.4)

The next command does the same thing as two lines above but from the positive, right hand side of the number line.

> map(q, [5, 4.5, 4.4, 4.2, 4.1, 4.05, 4.01, 4.001, 4.0001]);
[1, 0.5, 0.4, 0.2, 0.1, 0.05, 0.01, 0.001, 0.0001] \quad (1.5)

> plot(f(x), x=-2..10, y=-20..20);

\[ x = 4 \quad \text{vertical asymptote} \]

Horizontal Asymptote

The function below has a horizontal asymptote of x=2, because both the numerator and denominator have the same degree and the ratio of their leading coefficients is 2.
\[ f := x \to \frac{(2x^2 + 6x)}{(x^2 - 4)}; \]

\[ f := x \to \frac{2x^2 + 6x}{x^2 - 4} \quad (2.1) \]

\[ \text{factor}(f(x)); \]

\[ \frac{2x(x+3)}{(x-2)(x+2)} \quad (2.2) \]

**Question:** What are the vertical asymptotes?

\[ L := 10; \]

\[ \text{plot}([2, f(x)], x=-L..L, y=-L..L, \text{color=[red,blue]}); \]

\[ \text{The function may cross its horizontal asymptote.} \]

\[ A \text{ More Complicated Example} \]

\[ g := x \to \frac{3(x+3)x(x-4)}{(x+2)(x-2)(x-5)}; \]

\[ g := x \to \frac{3(x+3)x(x-4)}{(x+2)(x-2)(x-5)} \quad (2.3) \]

\[ W := 10; \quad H := 12; \]

Notice how the function crosses the horizontal asymptote \( x=3 \) in two places.

\[ \text{plot}([3, g(x)], x=-W..W, y=-H..H, \text{color=[red,blue]}, \text{numpoints=1000}); \]

\[ \text{It may cross its H.A. many times.} \]

\[ \text{▼ Slant Asymptote} \]

A slant asymptote occurs when the degree of the numerator is one greater than the degree of the
The equation of the slant asymptote is the quotient obtained from polynomial long division, or synthetic division.

\[ n := (x+5)(x-1)(x-8) \]
\[ d := (x-4)(x+4) \]
\[ f := \frac{(x+5)(x-1)(x-8)}{(x-4)(x+4)} \]
\[ x \to \frac{(x+5)(x-1)(x-8)}{(x-4)(x+4)} \]
\[ \text{quo}(n, d, x); \]
\[ x - 4 \]