3.6 Rational Functions

Def A <u>rational function</u> is a ratio or quotient of two polynomials.

$$f(x) = \frac{p(x)}{q(x)}$$

Ex. $f(x) = \dfrac{2x+5}{x-2}$

In addition to the polynomial graphing "tools" we developed in section 3.5, today we introduce three more:

1.) Vertical asymptotes
2.) Horizontal asymptote
3.) Slant asymptote

These are similar, and each function only has one or none.

1.) <u>Vertical asymptotes</u> are <u>vertical lines</u> which correspond with the zeros of the denominator polynomial.

Ex. $f(x) = \dfrac{2x+5}{x-2} \Rightarrow$ vert. asym.: $\boxed{x=2}$

Ex. $g(x) = \dfrac{x}{x^2-9} = \dfrac{x}{(x+3)(x-3)} \Rightarrow$ 2 vert. asym. $\boxed{x=-3}$ & $\boxed{x=3}$

<u>Note</u>: A rational function may have several vertical asymptotes, but the function will <u>NEVER</u> cross any of them.

2.) <u>Horizontal asymptote</u> occurs when $\deg(p(x)) \leq \deg(q(x))$

Ex. $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{4x^4-18}{2x^4+3x^2+2}$

case 1 Equal degrees

$y = \frac{4}{2} \Rightarrow \boxed{y=2}$ Just take the ratio of the leading coefficients of $p(x)$ & $q(x)$.

<u>case 2</u> $\boxed{\deg(p(x)) < \deg(q(x))}$

<u>Ex.</u> $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{100x + 50}{x^2 + 1} \Rightarrow$ H.A. $\boxed{y = 0}$

<u>why?</u> Even though $100x$ is big compared to x^2 for small values of x, when $x \to \pm\infty$, the denominator will be much, much bigger than the numerator, and the ratio will essentially be 0.

3.) <u>Slant asymptote</u> occurs when $\deg(p(x)) = \deg(q(x)) + 1$

<u>Ex.</u> $f(x) = \dfrac{2(x+2)(x-2)}{x-3} = \dfrac{2x^2-8}{x-3}$

Do polynomial long division:

$$
\begin{array}{r}
2x + 6 \\
x-3 \overline{\big) 2x^2 + 0x - 8} \\
-(2x^2 - 6x) \\
\hline
6x - 8 \\
-(6x - 18) \\
\hline
10
\end{array}
\qquad + \frac{10}{x-3}
$$

The quotient is the equation of the slant asymptote!

<u>Note:</u> when $\deg(p) > \deg(q) + 1$, you still get asymptotes, but we won't cover that case.

Vertical Asymptotes

The first three lines define a rational function:

```
> p := x-> 3*x-1: # numerator polynomial
> q := x-> x-4: # denominator polynomial
> f := x-> (3*x-1)/(x-4);
```

$$f := x \rightarrow \frac{3x-1}{x-4} \tag{1.1}$$

Approaching 4 from the left (from negative infinity)

The next command evaluates the polynomial p(x) at the x values given in the array below, notice that as we get closer to 4, the values change very little.

```
> map(p, [3, 3.5, 3.6, 3.8, 3.9, 3.95, 3.99, 3.999, 3.9999]);
```

$$[8, 9.5, 9.8, 10.4, 10.7, 10.85, 10.97, 10.997, 10.9997] \tag{1.2}$$

This next line evaluates the polynomial q(x) at the same x values as above. Notice how the values get very small and approach 0.

```
> map(q, [3, 3.5, 3.6, 3.8, 3.9, 3.95, 3.99, 3.999, 3.9999]);
```

$$[-1, -0.5, -0.4, -0.2, -0.1, -0.05, -0.01, -0.001, -0.0001] \tag{1.3}$$

Approaching 4 from the right (from positive infinity)

The next command does the same thing as two lines above but from the positive, right hand side of the number line.

```
> map(p, [5, 4.5, 4.4, 4.2, 4.1, 4.05, 4.01, 4.001, 4.0001]);
```

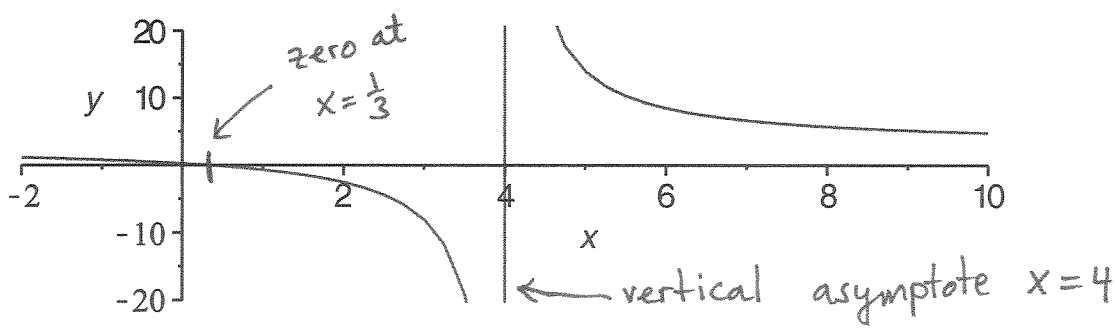
$$[14, 12.5, 12.2, 11.6, 11.3, 11.15, 11.03, 11.003, 11.0003] \tag{1.4}$$

The next command does the same thing as two lines above but from the positive, right hand side of the number line.

```
> map(q, [5, 4.5, 4.4, 4.2, 4.1, 4.05, 4.01, 4.001, 4.0001]);
```

$$[1, 0.5, 0.4, 0.2, 0.1, 0.05, 0.01, 0.001, 0.0001] \tag{1.5}$$

```
> plot(f(x), x=-2..10, y=-20..20);
```



Horizontal Asymptote

The function below has a horizontal asymptote of x=2, because both the numerator and denominator have the same degree and the ratio of their leading coefficients is 2.

```
> f := x -> (2*x^2 + 6*x)/(x^2-4);
```

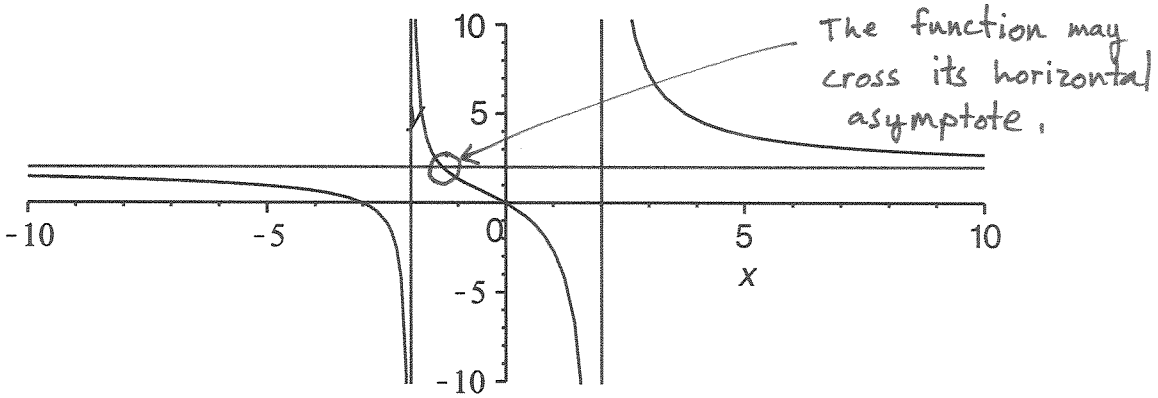
$$f := x \rightarrow \frac{2x^2 + 6x}{x^2 - 4} \tag{2.1}$$

```
> factor(f(x));
```

$$\frac{2x(x+3)}{(x-2)(x+2)} \tag{2.2}$$

Question: What are the vertical asymptotes?

```
> L := 10:
> plot({2,f(x)}, x=-L..L, y=-L..L, color=[red,blue]);
```



A More Complicated Example

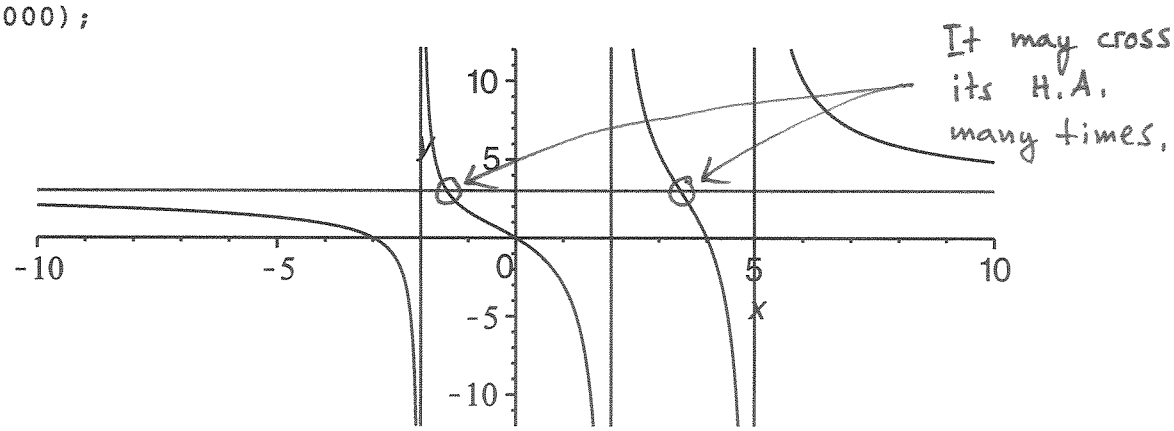
```
> g := x -> 3*(x+3)*x*(x-4)/((x+2)*(x-2)*(x-5));
```

$$g := x \rightarrow \frac{3(x+3)x(x-4)}{(x+2)(x-2)(x-5)} \tag{2.3}$$

```
> W := 10: H:= 12:
```

Notice how the function crosses the horizontal asymptote x=3 in two places.

```
> plot({3,g(x)}, x=-W..W, y=-H..H, color=[red,blue], numpoints=
  1000);
```



Slant Asymptote

A slant asymptote occurs when the degree of the numerator is *one* greater than the degree of the

denominator.
The equation of the slant asymptote is the quotient obtained from polynomial long division, or synthetic division.

```
> n := (x+5)*(x-1)*(x-8); # numerator polynomial
```

$$n := (x+5)\,(x-1)\,(x-8) \tag{3.1}$$

```
> d := (x-4)*(x+4); # denominator polynomial
```

$$d := (x-4)\,(x+4) \tag{3.2}$$

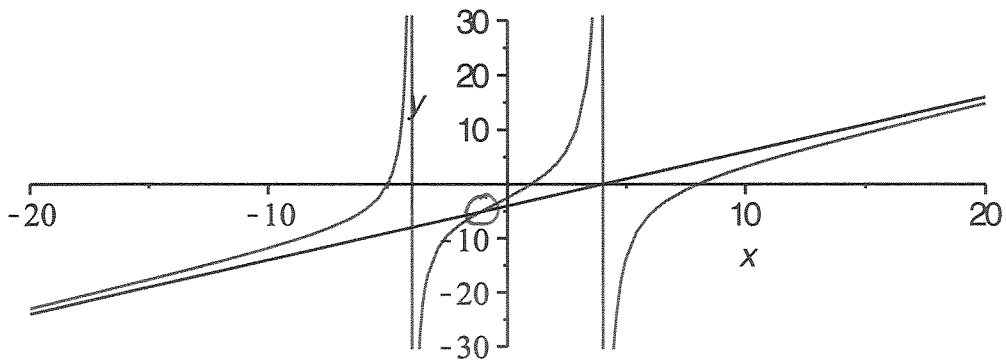
```
> f := x -> ((x+5)*(x-1)*(x-8))/((x-4)*(x+4));
```

$$f := x \rightarrow \frac{(x+5)\,(x-1)\,(x-8)}{(x-4)\,(x+4)} \tag{3.3}$$

```
> quo(n, d, x);
```

$$x-4 \tag{3.4}$$

```
> W := 20:  H := 30:
> plot({x-4,f(x)}, x=-W..W, y=-H..H, color=[red,blue]);
```




```
>
```