3.5 Graphing Polynomial Functions & Piecewise Functions

A polynomial function is a function that can be written with form:

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]

- **degree**
- **leading coefficient**
- **coefficients**

In this section we study the properties of polynomials which determine the shape of their graphs. First, let's review what we already know about linear (degree=1) polynomials and quadratic (degree=2) polynomials.

**Linear:** \( f(x) = a_1 x + a_0 = mx + b \)

- **slope**
- **y-intercept** (so named because \( f(0) = a_0 \))

\( a_1 < 0 \Rightarrow \text{decreasing or downward pointing line.} \)

\( a_1 > 0 \Rightarrow \text{increasing or upward "\" line.} \)

**Quadratic:** \( f(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c \)

- **concave down**
- **concave up**

\( a_2 < 0 \Rightarrow \text{concave down} \)

\( a_2 > 0 \Rightarrow \text{concave up} \)
How about polynomials of higher degree? There are a few general principles that the graphs of higher degree polynomials follow:

**Principle 1**

Degree - 1 = Maximum # of direction changes

For example:

\[ f(x) = x^3 - x = (x - 0)(x - 1)(x + 1) \]

Degree = 3

2 changes of direction

\[ g(x) = x^3 = (x - 0)(x - 0)(x - 0) = (x - 0)^3 \]

Degree = 3

0 changes in direction

We don't count this as a change in direction because the slope never changes from positive to negative.

\[ p(x) = x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x + 1)(x - 1)(x + 2)(x - 2) \]

Degree = 4

3 changes of direction

\[ q(x) = (x + 2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16 \]

Degree = 4
Principle 2
The sign of the leading coefficient and the evenness/oddness of the degree determine the polynomials' behavior at ±∞.

(Here the "..." represents wiggles or changes in direction.)

Principle 3
If a polynomial factors, then
1) factors raised to an odd power cross the x-axis.
2) "" "" "" even "" kiss "" .

Let $n \in \mathbb{N}$, then $z^n$ is even and $z^{n+1}$ is odd.

<table>
<thead>
<tr>
<th></th>
<th>$(x-a)^z$</th>
<th>$(x-a)^{z+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>$^a\rightarrow$</td>
<td>$^a\rightarrow$</td>
</tr>
<tr>
<td>even</td>
<td>$^a\rightarrow$</td>
<td>$^a\rightarrow$</td>
</tr>
</tbody>
</table>
Principle 4

The higher the power of a factor, the flatter the crossing or kissing becomes.

\[ x^3 = (x-0)^3 \]

\[ x^7 = (x-0)^7 \]

Crossings

\[ x^2 = (x-0)^2 \]

\[ x^{10} = (x-0)^{10} \]

Kissings

Example:

\[ f(x) = x^5 + x^4 - 2x^2 + x + 1 \] standard form

\[ f(x) = (x+1)^3 \cdot (x-1)^2 \] factored form

\[ f(x) \]
Piecewise Functions

cut the real number line \( \mathbb{R} \) into pieces, and define the function however you wish on each piece.

Example

\[
f(x) = \begin{cases} 
  x & x \geq 0 \\
-x & x < 0 
\end{cases}
\]

pieces of \( \mathbb{R} \).

function values

Q: What is the common name for \( f(x) \) above?

Example

\[
g(x) = \begin{cases} 
  2x + 1 & x < 1 \\
  4 & 1 \leq x < 2 \\
  3 & x \geq 2 
\end{cases}
\]

graph: