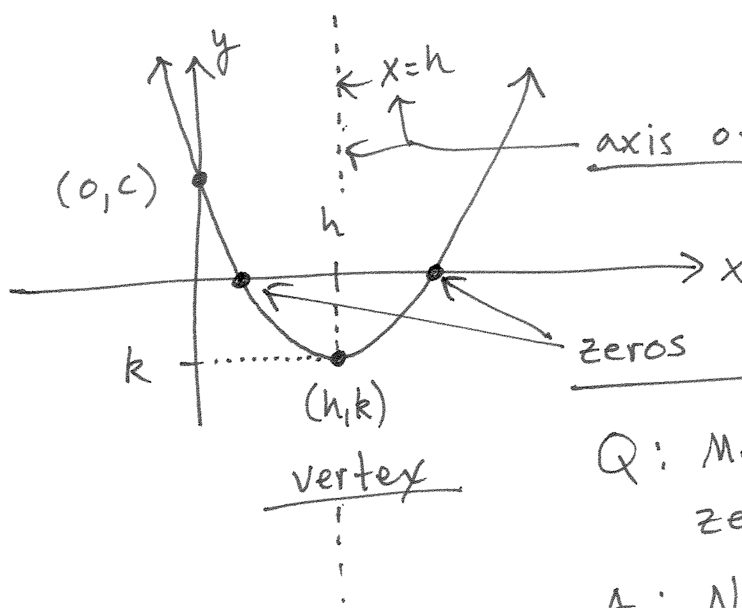
3.3 Parabolas : The Graphs of Quadratic Equations

Suppose you want to graph the quadratic equation $y = ax^2 + bx + c$, but you want to be able to do so easily. The secret lies in <u>completing the square.</u>

$$y = ax^2 + bx + c \xRightarrow{\text{complete the square}} y = a(x-h)^2 + k$$

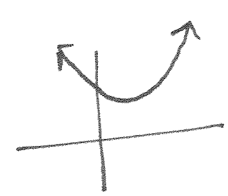
This second form of the quadratic equation is much easier to graph because if we let $x = h$, then we get: $y = a\underbrace{(h-h)^2}_{=0} + k = k$

Thus the point (h, k) lies on the graph.



Q: Must all parabolas have zeros?

A: No! e.g.



Q: How do we get <u>h</u> and <u>k</u>?

A: Complete the square on the general equation! $\longrightarrow$

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$a\left(x^2 + \frac{b}{a}x + \underbrace{\left(\frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right)}_{=0} + \frac{c}{a}\right) = 0 \quad \Leftarrow \left\{\begin{array}{l}\text{Completing}\\ \text{the square}\end{array}\right.$$

$$\left.\begin{array}{l}
a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2}\right] = 0 \\[2ex]
a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac}{4a^2} - \frac{b^2}{4a^2}\right] = 0 \\[2ex]
a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] = 0 \\[2ex]
a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0 \\[2ex]
a\left(x - \frac{-b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0
\end{array}\right\} \text{algebraic rearrangement}$$

$$y = a(x-h)^2 + k$$

So $\boxed{h = \frac{-b}{2a}}$ $\boxed{k = \frac{4ac - b^2}{4a}}$

The vertex of <u>any</u> parabola is given by:

$$\boxed{(h,k) = \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)}$$

Very handy to memorize and it's not in the text.

$y = a(x-h)^2 + k$

Two cases:

1.) $\underline{a>0}$ $\underline{\text{concave up}}$ 

2.) $\underline{a<0}$ $\underline{\text{concave down}}$ 

$\underline{\text{Ex.}}$ Put $y = \underset{a}{-3}x^2 \underset{b}{-6}x \underset{c}{+9}$ into $y=a(x-h)^2+k$ form,

$$h = \frac{-b}{2a} = \frac{-(-6)}{2(-3)} = \frac{6}{-6} = -1$$

$$k = \frac{4ac-b^2}{4a} = \frac{4(-3)9-36}{4(-3)} = \frac{\cancel{(-12)}(9)+3\cancel{(-12)}}{\cancel{(-12)}}$$

$$= 9+3 = 12$$

$$y = -3\left(x-(-1)\right)^2 + 12$$

$$y = -3(x+1)^2 + 12$$

Zeros?

$$0 = -3(x+1)^2 + 12$$

$$-12 = -3(x+1)^2$$

$$4 = (x+1)^2$$

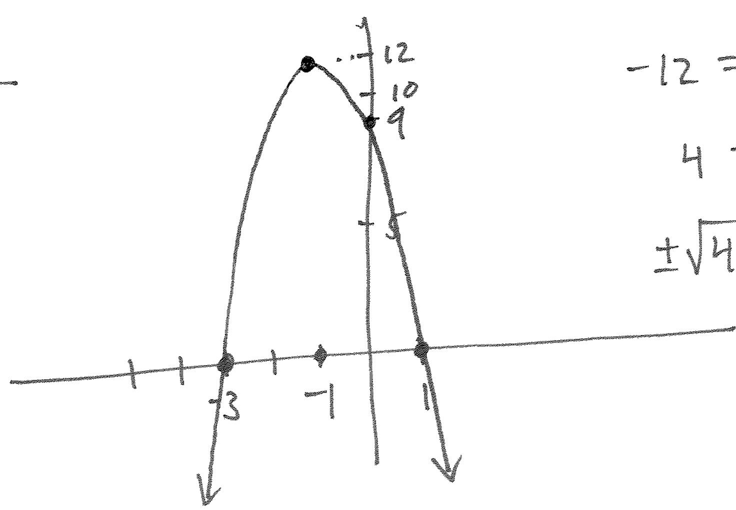
$$\pm\sqrt{4} = x+1$$

$$x = -1 \pm \sqrt{4}$$

$$x = -1 \pm 2$$

$$x = 1, -3$$

$\underline{\text{Graph it}}$



44. When an object is thrown upwards with a speed of 100 ft/sec., its height above ground is given by

$y(t) = -16t^2 + 100t$ where t is the time in seconds after it has been thrown.

a) At what time will the object reach its highest point?

$$y(t) = \underbrace{-16}_{a}t^2 + \underbrace{100}_{b}t + \underbrace{0}_{c}$$

$$h = \frac{-b}{2a} = \frac{-100}{2(-16)} = \frac{50}{16} = \boxed{3.125 \text{ s}}$$

b) How high will that point be (in feet)?

$$k = \frac{4ac - b^2}{4a} = \frac{4(-16)(0) - (100)^2}{4(-16)}$$

$$= \frac{-10{,}000}{-64}$$

$$= \boxed{156.25 \text{ ft}}$$