2.4 If $A = \begin{bmatrix}5&2\\-2&1\end{bmatrix}$ compute A^{-1}:

1. Augment with a 2x2 identity matrix.
2. Use row operations to transform augmented matrix so that its left-hand side is the 2x2 identity.
3. The right-hand side will be A^{-1}.

$$\left[\begin{array}{cc|cc}5&2&1&0\\-2&1&0&1\end{array}\right]^{R_1+2R_2}\left[\begin{array}{cc|cc}1&4&1&2\\-2&1&0&1\end{array}\right]_{R_2+2R_1}\left[\begin{array}{cc|cc}1&4&1&2\\0&9&2&5\end{array}\right]\tfrac{1}{9}R_2$$

$$\left[\begin{array}{cc|cc}1&4&1&2\\0&1&2/9&5/9\end{array}\right]^{R_1-4R_2}\left[\begin{matrix}1&0\\0&1\end{matrix}\right|\underbrace{\left.\begin{matrix}1/9&-2/9\\2/9&5/9\end{matrix}\right]}_{A^{-1}}$$

We can use A^{-1} to solve matrix equations involving A like so:

$$A\vec{x}=\vec{b} \iff \begin{bmatrix}5&2\\-2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix} \quad (*)$$

$$A^{-1}A\vec{x}=A^{-1}b \iff I\vec{x}=A^{-1}\vec{b} \iff \vec{x}=A^{-1}\vec{b} \iff \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1/9&-2/9\\2/9&5/9\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$

$$\iff \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1/9+-4/9\\2/9+10/9\end{bmatrix}=\begin{bmatrix}-1/3\\4/3\end{bmatrix}$$

This solves (*): $5x+2y=1$
$-2x+y=2$

check: $5(-1/3)+2(4/3)=\frac{3}{3}=1$ ✓
$-2(-1/3)+1(4/3)=\frac{6}{3}=2$ ✓

However, the real benefit to this method is that we can use A^{-1} to solve equation (*) for any vector $\vec{b}$!! →

For example if $\vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/9 & -2/9 \\ 2/9 & 5/9 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/9 + 0 \\ 6/9 + 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

check: $5\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = \frac{9}{3} = 3$ ✓

$-2\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = \frac{0}{3} = 0$ ✓

Note that we can also write $A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

<u>Question</u>: Is there a way to find a general formula for the inverse of a 2x2 matrix?

<u>Answer</u>: Yes. Let's use the above technique on the general 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a,b,c,d \in \mathbb{R}$.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & -\frac{bc}{a} + \frac{ad}{a} & -c/a & 1 \end{array}\right] \frac{a}{ad-b}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{-\frac{b}{a}R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{bc}{a(ad-bc)} + \frac{1}{a} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right| \underbrace{\left.\begin{array}{cc} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]}_{A^{-1}}$$

$$\boxed{\begin{array}{l} \text{So if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{array}}$$

Notice: A^{-1} is undefined if $ad - bc = 0$.

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & 4 & 2 \end{bmatrix}$, compute A^{-1}:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \tfrac{1}{2}R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{2} & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] -R_1+R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{2} & 0 & 0 \\ 0 & 2 & 2 & -\tfrac{1}{2} & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] -2R_2+R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{2} & 0 & 0 \\ 0 & 2 & 2 & -\tfrac{1}{2} & 1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} \tfrac{1}{2}R_2 \\ -\tfrac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\tfrac{1}{4} & \tfrac{1}{2} & 0 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 1 & -\tfrac{1}{2} \end{array}\right] -R_3+R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \tfrac{1}{4} & -\tfrac{1}{2} & \tfrac{1}{2} \\ 0 & 0 & 1 & -\tfrac{1}{2} & 1 & -\tfrac{1}{2} \end{array}\right]$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

To help you do your homework, go to

www.zweigmedia.com/RealWorld/

→ On Line Utilities → Pivot and Gauss-Jordan Tool

Then click on the link: "Click here for some detailed instructions."

You can use this tool to check your work, and to eliminate arithmetic mistakes.

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix} \qquad \text{compute } A^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 1 & 1 & 7 & 0 & 1 & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right] R_2 - R_1 \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & -4 & 5 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right] R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & -1 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right] (-1) \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right] R_3 + 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right] R_2 + R_3 \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -4 & 5 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right] R1 - 5R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -19 & 20 & -25 \\ 0 & 1 & 0 & 4 & -4 & 5 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right] R_1 - 2R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -25 & 26 & -33 \\ 0 & 1 & 0 & 4 & -4 & 5 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right]$$

Summary

To compute A^{-1}

1. Construct $[A \mid I]$
2. Use row operations to transform this to

   $[I \mid B]$ (if possible)

3. Then $B = A^{-1}$, however,
4. If we obtain a row of zeros to the left of the vertical line, A^{-1} does not exist.