1.4 Systems of Linear Equations

Def. A system of equations is a set of two or more equations in two or more variables.

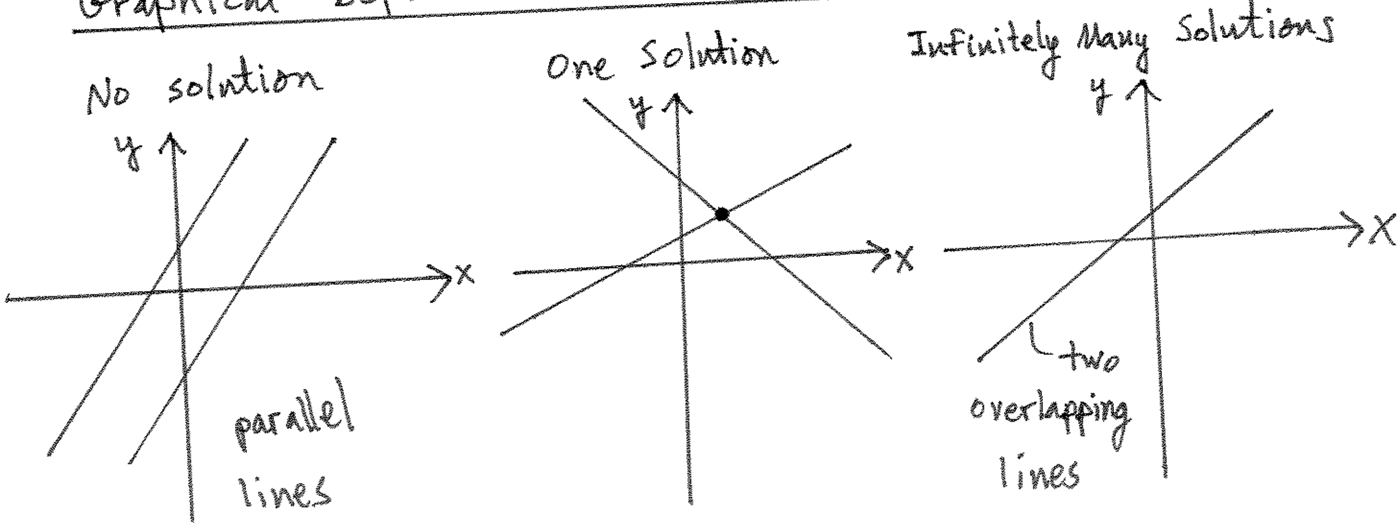
A solution is the set of all points which satisfy all equations in the system.

With any system of linear equations there are exactly three possibilities for the solution set:

1. No solution.
2. Exactly one solution.
3. Infinitely many solutions.

We are going to mostly stick to systems of linear equations in two variables e.g. x, y or systems of linear equations in three variables e.g. x, y and z.

Graphical Depiction of 3 Scenarios Above



Substitution Method of Solution

(2 Equations in 2 unknowns (variables))

Procedure	Example
	Solve: $\begin{cases} 2x + 3y = 4 \\ x - 2y = 3 \end{cases}$
1. Solve one of the equations for one of the variables in terms of the other.	1. $x - 2y = 3 \Rightarrow x = 2y + 3$
2. Substitute this expression into the other equation to give one equation in one unknown.	2. $2\overset{2y+3}{x} + 3y = 4$ $2(2y+3) + 3y = 4$
3. Solve this linear equation for the unknown	3. $4y + \underset{-6}{6} + 3y = \underset{-6}{4}$ $4y + 3y = -2$ $7y = -2$ $\boxed{y = \frac{-2}{7}}$
4. Substitute this solution into the equation in step 1 or into one of the original equations to solve for the other variable.	4. $x = 2\left(\frac{-2}{7}\right) + 3$ $x = \frac{-4}{7} + \frac{21}{7}$ $\boxed{x = \frac{17}{7}}$
5. Check the solution by substituting for x & y in both original equations.	5. $2\left(\frac{17}{7}\right) + 3\left(\frac{-2}{7}\right) = \frac{28}{7} = 4$ ✓ $\frac{17}{7} - 2\left(\frac{-2}{7}\right) = \frac{21}{7} = 3$ ✓

Elimination Method for Solving Systems

Procedure

1. If necessary, multiply one or both equations by a nonzero number that will make the coefficients of one of the variables identical, except perhaps signs.

2. Add or subtract the equations to eliminate one of the variables.

3. Solve for the variable in the resulting equation.

4. Substitute the solution into one of the original equations and solve for the other variable.

5. Check the solutions in both original equations.

Example

Solve the system $\begin{cases} 2x - 5y = 4 & (1) \\ x + 2y = 3 & (2) \end{cases}$

1. Multiply equation (2) by -2.

$$\begin{array}{rl} & 2x - 5y = 4 \\ + & -2x - 4y = -6 \\ \hline \end{array}$$

2. $0x - 9y = -2$

(eliminated x)

3. $\boxed{y = \frac{-2}{-9} = \frac{2}{9}}$

4. $2x - 5\left(\frac{2}{9}\right) = 4$

$$2x - \frac{10}{9} = 4$$

$$2x = \frac{10}{9} + \frac{36}{9}$$

$$2x = \frac{46}{9}$$

$$\boxed{x = \frac{23}{9}}$$

5. $2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) = \frac{36}{9} = 4$ ✓

$$\frac{23}{9} + 2\left(\frac{2}{9}\right) = \frac{27}{9} = 3 \ ✓$$

Example 1 Elimination

$$\begin{cases} 4x+3y=4 & (1) \\ 8x+6y=18 & (2) \end{cases}$$

Multiply eqn (1) by -2 and add to eqn. (2):

$$\begin{array}{r} -8x-6y=-8 \\ +\ 8x+6y=18 \\ \hline 0x+0y=10 \end{array}$$

(parallel lines)

This system is solved when $0=10$, which is never or impossible, so there are <u>no</u> <u>solutions</u>

Example 2

$$\begin{cases} 4x+3y=4 & (1) \\ 8x+6y=8 & (2) \end{cases}$$

Mult. eqn. (1) by -2:

$$\begin{array}{r} -8x-6y=-8 \\ +\ 8x+6y=8 \\ \hline 0x+0y=0 \end{array}$$

$\Rightarrow$ The equations are <u>dependent</u> i.e. the two equations are actually the same line

$\Rightarrow$ <u>infinitely many solutions</u>

Example 3

A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

solution: Let $x =$ amount of 12.5% solution (cc's)

$y =$ " " 5.0% " "

Recall: $\text{concentration} = \dfrac{\text{mass}}{\text{volume}} = \dfrac{\text{grams}}{\text{cc}}$ or maybe milligrams it does not matter

$$\begin{cases} x + y = 20 & (1) \\ .125x + .05y = .08(20) & (2) \end{cases}$$

I forgot this!

$100 \cdot$ eqn (2) $\Rightarrow$ $12.5x + 5y = 160$

Solve eqn. (1) for y: $y = -x + 20$

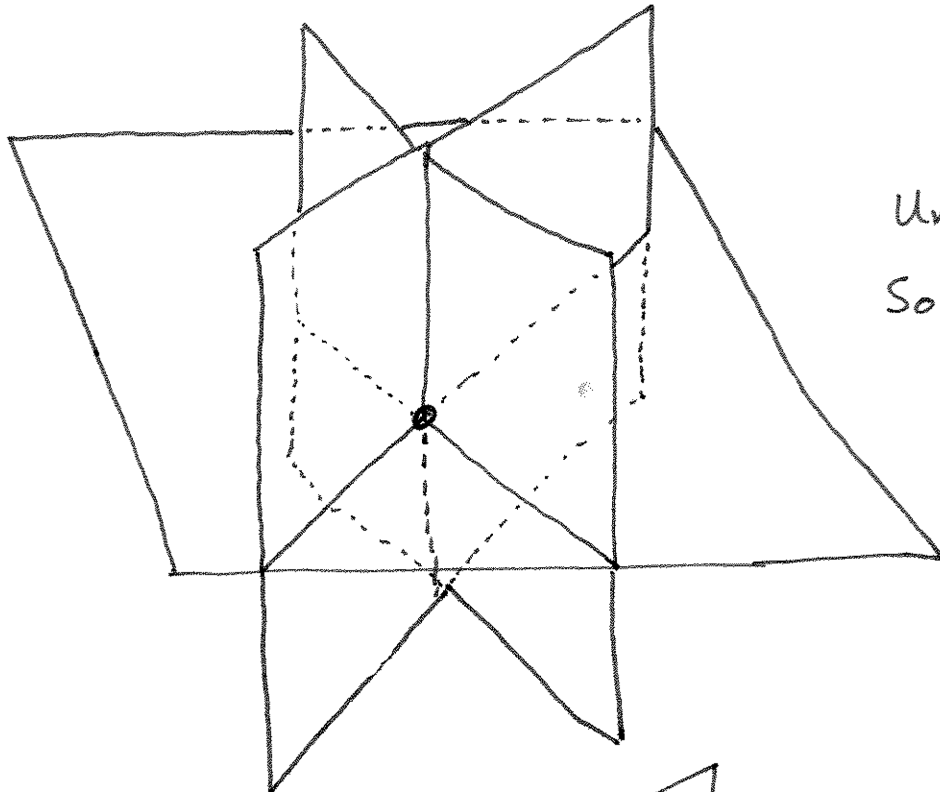
Plug into eqn. (2): $12.5x + 5(-x+20) = 160$

$\Rightarrow 7.5x + 100 = 160 \Rightarrow 7.5x = 60$

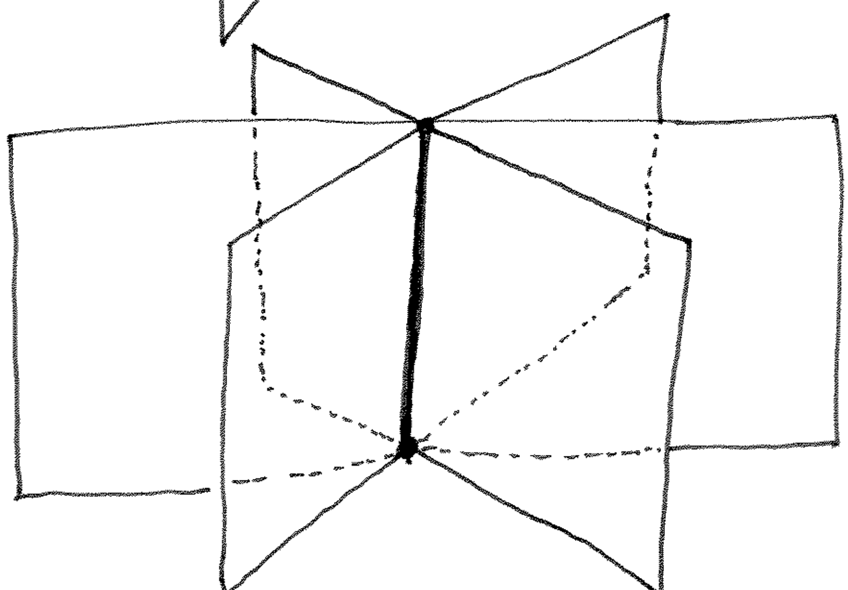
$\Rightarrow \frac{15}{2}x = 60 \Rightarrow x = \frac{120}{15} = 8$ So $\boxed{x=8}$

$8 + y = 20 \Rightarrow \boxed{y = 12}$

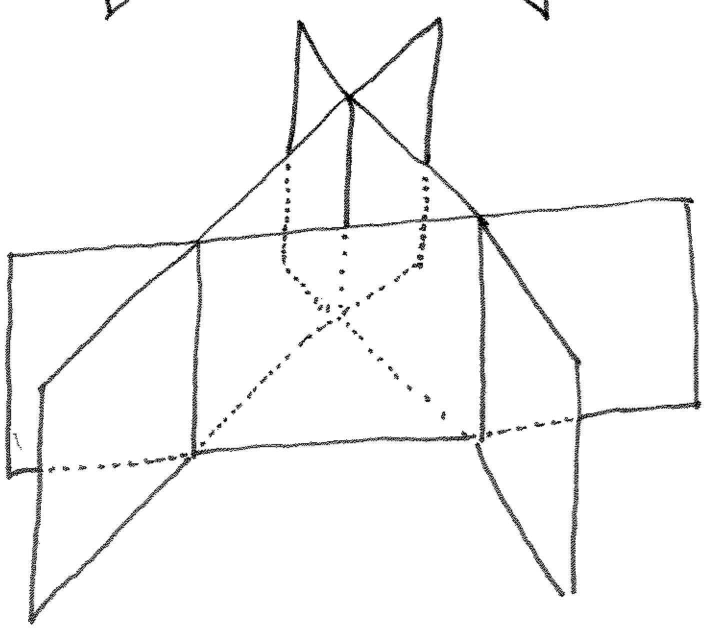
Three Equations in Three Variables



Unique Solution



Infinitely many solutions



No solution

Example 4 (problem 55)

$$\begin{aligned} 5x + y + 4z &= 18 \\ 6y &= 2z \\ 7y + z &= 10 \end{aligned} \Longrightarrow \begin{aligned} 5x + y + 4z &= 18 \quad (1) \\ 6y - 2z &= 0 \quad (2) \\ 7y + z &= 10 \quad (3) \end{aligned}$$

Mult. eqn (3) by 2, then add eqns. (2) & (3):

$$\begin{array}{rrr} & 6y - 2z & = 0 \\ + & 14y + 2z & = 20 \\ \hline & 20y & = 20 \end{array} \Longrightarrow \boxed{y = 1}$$

plug into eqn. (3):

$$7(1) + z = 10 \Longrightarrow \boxed{z = 3}$$

plug $y=1$ & $z=3$ into eqn. (1):

$$5x + 1 + 4(3) = 18$$

$$5x + 13 = 18$$

$$5x = 5$$

$$\boxed{x = 1}$$

Check:

(1) $5 \cdot 1 + 1 + 4 \cdot 3 = 18$ ✓

(2) $6 \cdot 1 - 2 \cdot 3 = 0$ ✓

(3) $7 \cdot 1 + 3 = 10$ ✓

Example (problem 66)

At a large school play, 1435 tickets were sold. A student ticket cost \$1.50 and an adult ticket cost \$5.00. If the ticket sales totalled \$3,552.50, how many of each type of ticket was sold?