1.4 Systems of Linear Equations

*Def.* A system of equations is a set of two or more equations in two or more variables.

A solution is the set of all points which satisfy all equations in the system.

With any system of linear equations there are exactly three possibilities for the solution set:

1. No solution.
2. Exactly one solution.
3. Infinitely many solutions.

We are going to mostly stick to systems of linear equations in two variables e.g. $x, y$ or systems of linear equations in three variables e.g. $x, y$ and $z$.

**Graphical Depiction of 3 Scenarios Above**

- **No solution**
  - Parallel lines

- **One Solution**
  - Single point of intersection

- **Infinitely Many Solutions**
  - Two overlapping lines

![Graphs](image.png)
Substitution Method of Solution

(2 Equations in 2 unknowns (variables))

Procedure

1. Solve one of the equations for one of the variables in terms of the other.

2. Substitute this expression into the other equation to give one equation in one unknown.

3. Solve this linear equation for the unknown.

4. Substitute this solution into the equation in step 1 or into one of the original equations to solve for the other variable.

5. Check the solution by substituting for x & y in both original equations.

Example

Solve:

\[ \begin{align*}
2x + 3y &= 4 \\
3x - 2y &= 3
\end{align*} \]

1. \( x - 2y = 3 \implies x = 2y + 3 \)

2. \( \frac{2y + 3}{3} \)
   \[ 2x + 3y = 4 \]
   \[ 2(2y + 3) + 3y = 4 \]
3. \( 4y + 6 + 3y = 4 \)
   \[ 4y + 3y = -2 \]
   \[ 7y = -2 \]
   \[ y = \frac{-2}{7} \]

4. \( x = 2\left(\frac{-2}{7}\right) + 3 \)
   \[ x = \frac{-4}{7} + \frac{21}{7} \]
   \[ x = \frac{17}{7} \]

5. \( 2\left(\frac{17}{7}\right) + 3\left(\frac{-2}{7}\right) = \frac{28}{7} = 4 \checkmark \)
   \[ \frac{17}{7} - 2\left(\frac{-2}{7}\right) = \frac{21}{7} = 3 \checkmark \)
**Elimination Method for Solving Systems**

**Procedure**

1. If necessary, multiply one or both equations by a nonzero number that will make the coefficients of one of the variables identical, except perhaps signs.

2. Add or subtract the equations to eliminate one of the variables.

3. Solve for the variable in the resulting equation.

4. Substitute the solution into one of the original equations and solve for the other variable.

5. Check the solutions in both original equations.

**Example**

Solve the system

\[
\begin{align*}
2x - 5y &= 4 \quad (1) \\
x + 2y &= 3 \quad (2)
\end{align*}
\]

1. Multiply equation (2) by 2:

\[
\begin{align*}
2x - 5y &= 4 \\
+ &-2x - 4y = -6 \\
\hline
0x - 9y &= -2
\end{align*}
\]

(eliminated \( x \))

2. \( 0x - 9y = -2 \)

3. \( y = \frac{-2}{-9} = \frac{2}{9} \)

4. \( 2x - 5\left(\frac{2}{9}\right) = 4 \)

\[
\begin{align*}
2x &= \frac{10}{9} + \frac{36}{9} \\
2x &= \frac{46}{9} \\
x &= \frac{23}{9}
\end{align*}
\]

5. \( 2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) = \frac{360}{9} = 4 \checkmark \)

\[
\begin{align*}
\frac{23}{9} + 2\left(\frac{2}{9}\right) &= \frac{27}{9} = 3 \checkmark
\end{align*}
\]
Example 1  Elimination

\[ \begin{align*}
4x + 3y &= 4 \quad (1) \\
8x + 6y &= 18 \quad (2)
\end{align*} \]

Multiply eqn (1) by \(-2\) and add to eqn (2):

\[ \begin{align*}
-8x - 6y &= -8 \\
+ 8x + 6y &= 18 \\
\hline
0x + 0y &= 10
\end{align*} \]

(parallel lines)

This system is solved when \(0 = 10\), which is never or impossible, so there are \underline{no solutions}.

Example 2

\[ \begin{align*}
4x + 3y &= 4 \quad (1) \\
8x + 6y &= 8 \quad (2)
\end{align*} \]

Multiply eqn (1) by \(-2\):

\[ \begin{align*}
-8x - 6y &= -8 \\
+ 8x + 6y &= 8 \\
\hline
0x + 0y &= 0
\end{align*} \]

\( \Rightarrow \) The equations are \underline{dependent}, i.e. the two equations are actually the same line

\( \Rightarrow \) infinitely many solutions
Example 3

A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

solution: Let \( x = \) amount of 12.5% solution (cc's)

\[ y = \] amount of 5.0% solution (cc's)

Recall: concentration = \( \frac{\text{mass}}{\text{volume}} = \frac{\text{grams}}{\text{cc}} \) or maybe milligrams

\[ \frac{\text{it does not matter}}{\text{matter}} \]

\[
\begin{cases}
  x + y = 20 \quad (1) \quad \text{I forgot this!} \\
  0.125x + 0.05y = 0.08(20) \quad (2)
\end{cases}
\]

100 eqn (2) \( \Rightarrow \) 12.5x + 5y = 160

solve eqn. (1) for \( y \): \( y = -x + 20 \)

plug into eqn. (2): \( 12.5x + 5(-x + 20) = 160 \)

\( \Rightarrow 7.5x + 100 = 160 \Rightarrow 7.5x = 60 \)

\( \Rightarrow \frac{15}{2}x = 60 \Rightarrow x = \frac{120}{15} = 8 \) so \( [x = 8] \)
Three Equations in Three Variables

Unique Solution

Infinitely Many Solutions

No Solution
Example 4  (problem 55)

\[ 5x + y + 4z = 18 \]  \hspace{1cm}  \[ 5x + y + 4z = 18 \quad (1) \]
\[ 6y = 2z \]  \hspace{1cm}  \[ 6y - 2z = 0 \quad (2) \]
\[ 7y + z = 10 \]  \hspace{1cm}  \[ 7y + z = 10 \quad (3) \]

Multiply eqn (3) by 2, then add eqns (2) \& (3):

\[
\begin{align*}
6y - 2z &= 0 \\
+ 14y + 2z &= 20 \\
20y &= 20 \\
\hline
y &= 1
\end{align*}
\]

plug into eqn. (3):

\[ 7(1) + z = 10 \quad \Rightarrow \quad \boxed{z = 3} \]

plug \( y = 1 \) \& \( z = 3 \) into eqn. (1):

\[ 5x + 1 + 4(3) = 18 \]
\[ 5x + 13 = 18 \]
\[ 5x = 5 \]
\[ \boxed{x = 1} \]

Check:

(1) \[ 5 \cdot 1 + 1 + 4 \cdot 3 = 18 \checkmark \]
(2) \[ 6 \cdot 1 - 2 \cdot 3 = 0 \checkmark \]
(3) \[ 7 \cdot 1 + 3 = 10 \checkmark \]
Example (problem 66)

At a large school play, 1435 tickets were sold. A student ticket cost $1.50 and an adult ticket cost $5.00. If the ticket sales totalled $3,552.50, how many of each type of ticket was sold?