UNIT 8A

Growth: Linear versus Exponential: We contrast the two basic forms of growth: *linear growth* by the same absolute amount in any time period and *exponential growth* by the same relative amount in any time period. We also explore the remarkable effects of the repeated doublings that characterize exponential growth.

UNIT 8B

Doubling Time and Half-Life: Exponential growth is characterized by a constant *doubling time,* while exponential decay is characterized by a constant *half-life.* We explore both ideas and some of their many applications.

UNIT 8C

Real Population Growth: Real population growth is neither strictly linear nor strictly exponential. We study both the reasons for and the limits to population growth, which have important implications for the future of the human race.

UNIT 8D

Logarithmic Scales: Earthquakes, Sounds, and Acids: We study the meaning and use of three important *logarithmic scales:* the magnitude scale for earthquakes, the decibel scale for sound, and the pH scale for acidity.

8

> The greatest shortcoming of the human race is (its) inability to understand the exponential function.
>
> —Albert A. Bartlett,
> Professor of Physics,
> University of Colorado

Exponential Astonishment

World population is currently growing by some 75 million people per year—enough to fill an entire new United States in only four years. A growing population means new challenges for our species, which we can meet only if we understand this growth. In this chapter, we will investigate the mathematical laws of growth—specifically, exponential growth. We will focus on what we call *exponential astonishment:* the intuition-defying reality of exponential growth. Although we will emphasize population growth, we will also study many other important topics, including the decay of waste from nuclear power plants, the depletion of natural resources, and the environmental effects of acid rain.

ACTIVITY Towers of Hanoi

Use this activity to gain a sense of the kinds of problems this chapter will enable you to study. Try working through it before you begin the chapter, then return to it after you've learned the chapter material.

The game called *Towers of Hanoi* consists of three pegs and a set of disks of varying sizes. Each disk has a hole in its center so that it can be moved from peg to peg. The game begins with all the disks stacked on one peg in order of decreasing size (see photo). The object of the game is to move the entire stack of disks to a different peg, following two rules:

Rule 1. Only one disk can be moved at a time.
Rule 2. A larger disk can never be placed on top of a smaller disk.

You can easily find or make a version of the Towers of Hanoi game; many Web sites have online simulations of the game, or you can make your own version by cutting disks out of cardboard. Play the game with seven disks, looking for the most efficient strategy for moving the disks. Once you have found the best strategy, answer the following questions.

1. You can view the game as a series of goals. The first goal is to end up with 1 disk on another peg, the second goal is to end up with 2 disks on another peg, and so on, until all the disks are on another peg. The first goal requires only one move: taking the smallest (top) disk and moving it to a different peg. The second goal then requires two more moves: first moving the second-smallest disk to the empty peg and then putting the smallest disk on top of it. Complete the following table as you continue the game.

Goal	Moves Required for This Step	Total Moves in Game So Far
1 disk on another peg	1	1
2 disks on another peg	2	$1 + 2 = 3$
3 disks on another peg		
4 disks on another peg		
5 disks on another peg		
6 disks on another peg		
7 disks on another peg		

2. Look at the patterns in the table. Find general formulas for the second and third columns after n steps. Confirm that your formulas give the correct results for all the entries in the table. (*Hint:* If you are using the most efficient strategy, both formulas will involve powers of 2, with n appearing in the exponent.)

3. Use the formula for the total moves in the game (column 3) to predict the total number of moves required to complete the game with 10 disks (rather than 7).

4. The game is related to a legend claiming that, at the beginning of the world, the Brahma put three large diamond needles on a brass slab in the Great Temple, and placed 64 disks of solid gold on one needle; the disks were arranged in order of decreasing size, just like the disks in the Towers of Hanoi game. Working in shifts, day and night, the temple priests moved the golden disks according to the two rules of the game. How many total moves are required to move the entire set of 64 disks to another needle?

5. The legend holds that upon completion of the task of moving all 64 disks, the temple will crumble and the world will come to an end. Assume that the priests

can move very fast, so they move one disk each second. Based on your answer to question 4, how many years will it take to move the entire stack of 64 disks? If the legend is true, do we have anything to worry about right now? (Useful data: Scientists estimate the current age of the universe to be about 14 billion years.)

6. The way in which the number of moves required for each step increases is an example of exponential growth—the topic of this chapter. Briefly comment on what this game illustrates about the nature of exponential growth.

UNIT 8A Growth: Linear versus Exponential

Imagine two communities, Straightown and Powertown, each with an initial population of 10,000 people (Figure 8.1). Straightown grows at a constant rate of 500 people per year, so its population reaches 10,500 after 1 year, 11,000 after 2 years, 11,500 after 3 years, and so on. Powertown grows at a constant rate of 5 *percent* per year. Because 5% of 10,000 is 500, Powertown's population also reaches 10,500 after 1 year. In the second year, however, Powertown's population increases by 5% of 10,500, which is 525, to 11,025. In the third year, Powertown's population increases by 5% of 11,025, or by 551 people. Figure 8.1 contrasts the populations of the two towns over a 45-year period. Note that Powertown's population rises ever more steeply and quickly outpaces Straightown's.

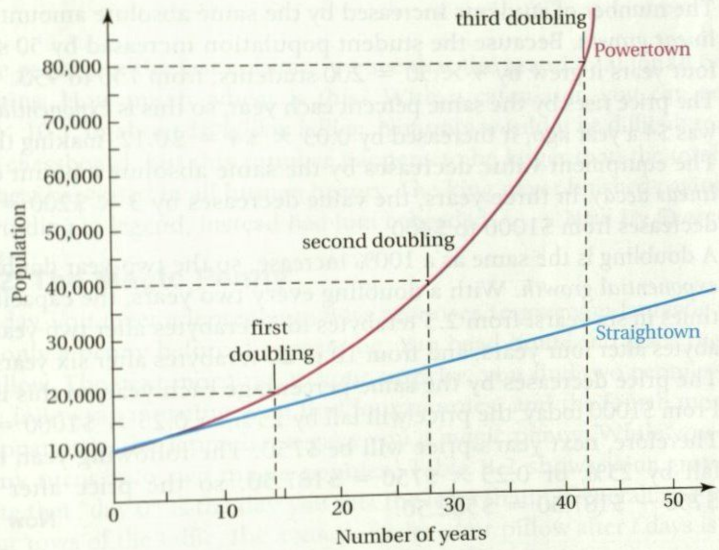


FIGURE 8.1 Straightown grows linearly, while Powertown grows exponentially.

Straightown and Powertown illustrate two fundamentally different types of growth. Straightown grows by the same *absolute* amount—500 people—each year, which is characteristic of **linear growth.** In contrast, Powertown grows by the same *relative* amount—5%—each year, which is characteristic of **exponential growth.**

> **TWO BASIC GROWTH PATTERNS**
>
> **Linear growth** occurs when a quantity grows by the same *absolute* amount in each unit of time.
>
> **Exponential growth** occurs when a quantity grows by the same *relative* amount—that is, by the same *percentage*—in each unit of time.

The terms *linear* and *exponential* can also be applied to quantities that decrease with time. For example, *exponential decay* occurs when a quantity decreases by the same relative amount in each unit of time. In the rest of this unit, we will explore the surprising properties of exponential growth.

Example 1 Linear or Exponential?

In each of the following situations, state whether the growth (or decay) is linear or exponential, and answer the associated questions.

a. The number of students at Wilson High School has increased by 50 in each of the past four years. If the student population was 750 four years ago, what is it today?
b. The price of milk has been rising 3% per year. If the price of a gallon of milk was \$4 a year ago, what is it now?
c. Tax law allows you to depreciate the value of your equipment by \$200 per year. If you purchased the equipment three years ago for \$1000, what is its depreciated value today?
d. The memory capacity of state-of-the-art computer hard drives is doubling approximately every two years. If a company's top-of-the-line drive holds 2.5 terabytes today, what will it hold in six years?
e. The price of high-definition TV sets has been falling by about 25% per year. If the price is \$1000 today, what can you expect it to be in two years?

Solution

a. The number of students increased by the same absolute amount each year, so this is *linear growth*. Because the student population increased by 50 students per year, in four years it grew by $4 \times 50 = 200$ students, from 750 to 950.
b. The price rises by the same percent each year, so this is *exponential growth*. If the price was \$4 a year ago, it increased by $0.03 \times \$4 = \$0.12$, making the price \$4.12.
c. The equipment value decreases by the same absolute amount each year, so this is *linear decay*. In three years, the value decreases by $3 \times \$200 = \$600$, so the value decreases from \$1000 to \$400.
d. A doubling is the same as a 100% increase, so the two-year doubling time represents *exponential growth*. With a doubling every two years, the capacity will double three times in six years: from 2.5 terabytes to 5 terabytes after two years, from 5 to 10 terabytes after four years, and from 10 to 20 terabytes after six years.
e. The price decreases by the same percentage each year, so this is *exponential decay*. From \$1000 today, the price will fall by 25%, or $0.25 \times \$1000 = \$250$, in one year. Therefore, next year's price will be \$750. The following year, the price will again fall by 25%, or $0.25 \times \$750 = \$187.50$, so the price after two years will be $\$750 - \$187.50 = \$562.50$.

Now try Exercises 9–16.

The Impact of Doublings

Look again at the graph of Powertown's population in Figure 8.1. After about 14 years, the original population has doubled to 20,000. In the next 14 years, it doubles again to 40,000. It then doubles again, to 80,000, 14 years after that. This type of repeated *doubling*, in which each doubling occurs in the same amount of time, is a hallmark of exponential growth.

The time it takes for each doubling depends on the rate of the exponential growth. In Unit 8B, we'll see how the doubling time depends on the percentage growth rate. Here, we'll explore three parables that show how doublings make exponential growth so very different from linear growth.

Parable 1: From Hero to Headless in 64 Easy Steps

Legend has it that, when chess was invented in ancient times, a king was so enchanted that he said to the inventor, "Name your reward."

"If you please, king, put one grain of wheat on the first square of my chessboard," said the inventor. "Then, place two grains on the second square, four grains on the third square, eight grains on the fourth square, and so on." The king gladly agreed, thinking the man a fool for asking for a few grains of wheat when he could have had gold or jewels. But let's see how it adds up for the 64 squares on a chessboard.

Table 8.1 shows the calculations. Each square gets twice as many grains as the previous square, so the number of grains on any square is a power of 2. The third column shows the total number of grains up to each point, and the last column gives a simple formula for the total number of grains.

TABLE 8.1

Square	Grains on This Square	Total Grains Thus Far	Formula for Total Grains
1	$1 = 2^0$	1	$2^1 - 1$
2	$2 = 2^1$	$1 + 2 = 3$	$2^2 - 1$
3	$4 = 2^2$	$3 + 4 = 7$	$2^3 - 1$
4	$8 = 2^3$	$7 + 8 = 15$	$2^4 - 1$
5	$16 = 2^4$	$15 + 16 = 31$	$2^5 - 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
64	2^{63}		$2^{64} - 1$

From the pattern in the last column, we see that the grand total for all 64 squares is $2^{64} - 1$ grains. How much wheat is this? With a calculator, you can confirm that $2^{64} = 1.8 \times 10^{19}$, or about 18 *billion billion*. Not only would it be difficult to fit so many grains on a chessboard, but this number happens to be larger than the total number of grains of wheat harvested in all human history. The king never finished paying the inventor and, according to legend, instead had him beheaded. **Now try Exercises 17–20.**

Parable 2: The Magic Penny

One lucky day, you meet a leprechaun who promises to give you fantastic wealth, but hands you only a penny before disappearing. You head home and place the penny under your pillow. The next morning, to your surprise, you find two pennies under your pillow. The following morning, you find four pennies, and the fourth morning, eight pennies. Apparently, the leprechaun gave you a *magic* penny: While you sleep, each magic penny turns into *two* magic pennies. Table 8.2 shows your growing wealth. Note that "day 0" is the day you met the leprechaun. Generalizing from the first four rows of the table, the amount under your pillow after t days is

$$\$0.01 \times 2^t$$

TABLE 8.2

Day	Amount Under Pillow
0	$\$0.01 \times 2^0 = \$0.01$
1	$\$0.01 \times 2^1 = \$0.02$
2	$\$0.01 \times 2^2 = \$0.04$
3	$\$0.01 \times 2^3 = \$0.08$
4	$\$0.01 \times 2^4 = \$0.16$
$\vdots$	$\vdots$
t	$\$0.01 \times 2^t$

We can use this formula to figure out how long it will be until you have fantastic wealth. After $t = 9$ days, you'll have $\$0.01 \times 2^9 = \$5.12$, which is barely enough to buy lunch. But by the end of a month, or $t = 30$ days, you'll have $\$0.01 \times 2^{30} = \$10{,}737{,}418.24$. That is, you'll be a millionaire within a month, and you'll need a much larger pillow! In fact, if your magic pennies keep doubling, by the end of just 51 days you'll have $\$0.01 \times 2^{51} \approx \$22.5$ trillion, which is more than enough to pay off the national debt of the United States. **Now try Exercises 21–24.**

Parable 3: Bacteria in a Bottle

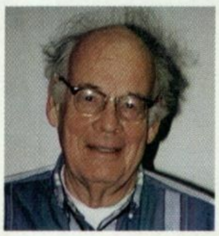
For our third parable, we turn to a story with relevance to the future of the human race. Suppose you place a single bacterium in a bottle at 11:00 a.m. It grows and at

11:01 divides into two bacteria. These two bacteria each grow and at 11:02 divide into four bacteria, which grow and at 11:03 divide into eight bacteria, and so on.

Now, suppose the bacteria continue to double every minute, and the bottle is full at 12:00. You may already realize that the number of bacteria at this point must be 2^{60} (because they doubled every minute for 60 minutes), but the important fact is that we have a bacterial disaster on our hands: Because the bacteria have filled the bottle, the entire bacterial colony is doomed. Let's examine this tragedy in greater detail, by asking a few questions about the demise of the colony.

BY THE WAY

The bacteria in a bottle parable was developed by Professor Albert A. Bartlett, University of Colorado, who has given more than 1670 lectures on the lessons of exponential growth around the country over the past 40 years.



- *Question 1:* If the tragedy occurred because the bottle was full at 12:00, when was the bottle *half*-full?

  Answer: Because it took one hour to fill the bottle, many people guess that it was half-full after a half-hour, or at 11:30. However, because the bacteria double in number every minute, they must also have doubled during the last minute, which means the bottle went from being half-full to full during the final minute. That is, the bottle was half-full at 11:59, just 1 minute before the disaster.

- *Question 2:* Imagine that *you* are a mathematically sophisticated bacterium, and at 11:56 you recognize the impending disaster. You immediately jump on your soapbox and warn that unless your fellow bacteria slow their growth dramatically, the end is just 4 minutes away. Will anyone believe you?

  Answer: You've done your mathematics correctly; the bottle will indeed be full in just 4 minutes. But how will it appear to others who have not done the calculations? As we've already seen, the bottle will be half-full at 11:59. Continuing to work backward through the doublings each minute, we find that it will be $\frac{1}{4}$ full at 11:58, $\frac{1}{8}$ full at 11:57, and $\frac{1}{16}$ full at 11:56. Therefore, if your fellow bacteria look around the bottle at 11:56, they'll see that only $\frac{1}{16}$ of the bottle has been used. In other words, the amount of unused space is *15 times* the amount of used space. You are asking them to believe that, in just the next 4 minutes, they'll fill 15 times as much space as they did in their entire 56-minute history. Unless they do the mathematics for themselves, they are unlikely to take your warnings seriously. Figure 8.2 shows the situation graphically. Note that the bottle remains nearly empty for most of the 60 minutes, but the continued doublings fill it rapidly in the final 4 minutes.

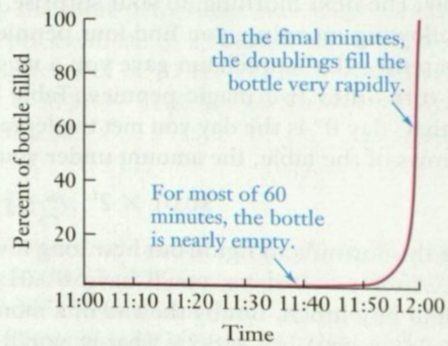


FIGURE 8.2 The population of the bacteria in the bottle

- *Question 3:* It's 11:59 and, with the bottle half-full, your fellow bacteria are finally taking your warnings seriously. They quickly start a space program, sending little bacterial spaceships out into the lab in search of new bottles. Thankfully, they discover three more bottles (making a total of four, including the one already occupied). Working quickly, they initiate a mass migration by packing bacteria onto spaceships and sending them to the new bottles. They successfully distribute the population evenly among the four bottles, just in time to avert the disaster. Given that they now have four bottles rather than just one, how much time have they gained for their civilization?

Answer: Because it took one hour to fill one bottle, you might guess that it would take four hours to fill four bottles. But remember that the bacterial population continues to *double* each minute. If there are enough bacteria to fill one bottle at 12:00, there will be enough to fill two bottles by 12:01 and four bottles by 12:02. The discovery of three new bottles gives them only 2 additional minutes.

Facts do not cease to exist because they are ignored.

—Aldous Huxley

- *Question 4:* Suppose the bacteria continue their space program, constantly looking for more bottles. Is there any hope that further discoveries will allow the colony to continue its exponential growth?

Answer: Let's do some calculations. After n minutes, the bacterial population is 2^n. For example, it is $2^0 = 1$ when the first bacterium starts the colony at 11:00, $2^1 = 2$ at 11:01, $2^2 = 4$ at 11:02, and so on. There are 2^{60} bacteria when the first bottle fills at 12:00, and 2^{62} bacteria when four bottles are full at 12:02. Suppose that, somehow, the bacteria managed to keep doubling every minute until 1:00. By that time, the number of bacteria would be 2^{120}, because it has been 120 minutes since the colony began. Now, we must figure out how much space they'd require for this population.

The smallest bacteria measure approximately 10^{-7} m (0.1 micrometer) across. If we assume that the bacteria are roughly cube-shaped, the volume of a single bacterium is

$$(10^{-7}\,\text{m})^3 = 10^{-21}\,\text{m}^3$$

Therefore, the colony of 2^{120} bacteria would occupy a total volume of

$$2^{120} \times 10^{-21}\,\text{m}^3 \approx 1.3 \times 10^{15}\,\text{m}^3$$

With this volume, the bacteria would cover the entire surface of the Earth in a layer more than 2 meters deep! (See Exercise 27 to calculate this result for yourself.)

In fact, if the doublings continued for just $5\frac{1}{2}$ hours, the volume of bacteria would exceed the volume of the entire universe (see Exercise 28). Needless to say, this cannot happen. The exponential growth of the colony cannot possibly continue for long, no matter what technological advances might be imagined. **Now try Exercises 25–28.**

Time Out to Think

Some people have suggested that we could find room for an exponentially growing human population by colonizing other planets in our solar system. Is this possible?

Doubling Lessons

The three parables reveal at least two key lessons about the repeated doublings that arise with exponential growth. First, if you look back at Table 8.1, you'll notice that the number of grains on each square is nearly equal to the total number of grains on all previous squares combined. For example, the 16 grains on the fifth square are 1 more than the total of 15 grains on the first four squares combined.

Second, all three parables show quantities growing to impossible proportions. We cannot possibly fit all the wheat harvested in world history on a chessboard; $22 trillion worth of pennies won't fit under your pillow, and it is far more than the number of pennies ever produced; and a colony of bacteria can't possibly fill the universe.

KEY FACTS ABOUT EXPONENTIAL GROWTH

- Exponential growth leads to repeated doublings. With each doubling, the amount of increase is approximately equal to the *sum* of all preceding doublings.
- Exponential growth cannot continue indefinitely. After only a relatively small number of doublings, exponentially growing quantities reach impossible proportions.

QUICK QUIZ

Choose the best answer to each of the following questions. Explain your reasoning with one or more complete sentences.

1. A town's population increases in one year from 100,000 to 110,000. If the population is growing *linearly*, at a steady rate, then at the end of a second year it will be

   a. 110,000. b. 120,000. c. 121,000.

2. A town's population increases in one year from 100,000 to 110,000. If the population is growing *exponentially* at a steady rate, then at the end of a second year it will be

   a. 110,000. b. 120,000. c. 121,000.

3. The balance owed on your credit card doubles from $1000 to $2000 in 6 months. If your balance is growing *exponentially*, how much longer will it be until it reaches $4000?

   a. 6 months b. 12 months c. 18 months

4. The number of songs in your iPod has risen from 200 to 400 in 3 months. If the number of songs is increasing *linearly*, how much longer will it be until you have 800 songs?

   a. 3 months

   b. 6 months

   c. $1\frac{1}{2}$ months

5. Which of the following is an example of *exponential decay?*

   a. The population of a rural community is falling by 100 people per year.

   b. The price of gasoline is falling by $0.02 per week.

   c. Government support for education is falling 1% per year.

6. On a chessboard with 64 squares, you place 1 penny on the first square, 2 pennies on the second square, 4 pennies on the third square, and so on. If you could follow this pattern to fill the entire board, about how much money would you need in total?

   a. about $1.28 b. about $500,000

   c. about 10,000 times as much as the current U.S. federal debt

7. At 11:00 you place a single bacterium in a bottle, and at 11:01 it divides into 2 bacteria, which at 11:02 divide into 4 bacteria, and so on. How many bacterial will be in the bottle at 11:30?

   a. 2×30 b. 2^{30} c. 2×10^{30}

8. Consider the bacterial population described in Exercise 7. How many more bacteria are in the bottle at 11:31 than at 11:30?

   a. 30 b. 2^{30} c. 2^{31}

9. Consider the bacterial population described in Exercise 7. If the bacteria occupy a volume of 1 cubic meter at 12:02 and continue their exponential growth, when will they occupy a volume of 2 cubic meters?

   a. 12:03 b. 12:04 c. 1:02

10. Which of the following is *not* true of any exponentially growing population?

    a. With every doubling, the population increase is nearly equal to the total increase from all previous doublings.

    b. The steady growth makes it easy to see any impending crisis long before the crisis becomes severe.

    c. The exponential growth must eventually stop.

Exercises 8A

REVIEW QUESTIONS

1. Describe the basic differences between linear growth and exponential growth.
2. Briefly explain how repeated doublings characterize exponential growth. Describe the impact of doublings, using the chessboard or magic penny parable.
3. Briefly summarize the story of the bacteria in the bottle. Be sure to explain the answers to the four questions asked in the text, and describe why the answers are surprising.
4. Explain the meaning of the two key facts about exponential growth given at the end of this unit. Then create your own example of exponential growth and describe the influence and impact of repeated doubling.

DOES IT MAKE SENSE?

Decide whether each of the following statements makes sense (or is clearly true) or does not make sense (or is clearly false). Explain your reasoning.

5. Money in a bank account earning compound interest at an annual percentage rate of 3% is an example of exponential growth.
6. Suppose you had a magic bank account in which your balance doubled each day. If you started with just $1, you'd be a millionaire in less than a month.
7. A small town that grows exponentially can become a large city in just a few decades.
8. Human population has been growing exponentially for a few centuries, and we can expect this trend to continue forever in the future.

BASIC SKILLS & CONCEPTS

9–16: Linear or Exponential? State whether the growth (or decay) is linear or exponential, and answer the associated question.

9. The population of MeadowView is increasing at a rate of 325 people per year. If the population is 2500 today, what will it be in four years?
10. The population of Winesburg is increasing at a rate of 3% per year. If the population is 75,000 today, what will it be in three years?
11. During the 1999 episode of hyperinflation in Brazil, the price of food increased at a rate of 30% per month. If your food bill was R$100 one month during this period, what was it three months later? (R$ is the symbol for the *real*, Brazil's unit of currency.)
12. The price of a gallon of gasoline is increasing by 3¢ per week. If the price is $3.10 per gallon today, what will it be in ten weeks?
13. The price of computer memory is decreasing at a rate of 14% per year. If a memory chip costs $50 today, what will it cost in three years?
14. The value of your car is decreasing by 10% per year. If the car is worth $12,000 today, what will it be worth in two years?
15. The value of your house is increasing by $2000 per year. If it is worth $100,000 today, what will it be worth in five years?
16. The value of your house is decreasing by 7% per year. If it is worth $250,000 today, what will it be worth in three years?

17–20: Chessboard Parable. Use the chessboard parable presented in the text. Assume that each grain of wheat weighs 1/7000 pound.

17. How many grains of wheat should be placed on square 16 of the chessboard? Find the total number of grains and their total weight (in pounds) at this point.
18. How many grains of wheat should be placed on square 32 of the chessboard? Find the total number of grains and their total weight (in pounds) at this point.
19. What is the total weight of all the wheat when the chessboard is full?
20. According to the U.S. Department of Agriculture, the current world harvest of all wheat, rice, and corn is less than 2 billion tons per year. How does this total compare to the weight of the wheat on the chessboard? (*Hint*: 1 ton = 2000 pounds.)

21–24: Magic Penny Parable. Use the magic penny parable presented in the text.

21. How much money would you have after 22 days?
22. Suppose that you stacked the pennies after 22 days. How high would the stack rise, in kilometers? (*Hint*: Find a few pennies and a ruler.)
23. How many days would elapse before you had a total of over $1 billion? (*Hint*: Proceed by trial and error.)
24. Suppose that you could keep making a single stack of the pennies. After how many days would the stack be long enough to reach the nearest star (beyond the Sun), which is about 4.3 light-years (4.0×10^{13} km) away? (*Hint*: Proceed by trial and error.)

25–26: Bacteria in a Bottle Parable. Use the bacteria parable presented in the text.

25. How many bacteria are in the bottle at 11:50? What fraction of the bottle is full at that time?
26. How many bacteria are in the bottle at 11:15? What fraction of the bottle is full at that time?
27. **Knee-Deep in Bacteria.** The total surface area of Earth is about 5.1×10^{14} m². Assume that the bacteria continued their doublings for two hours (as discussed in the text), at which point they were distributed uniformly over Earth's surface. How deep would the bacterial layer be? Would it be knee-deep, more than knee-deep, or less than knee-deep? (*Hint*: Remember that a volume divided by an area gives a depth.)
28. **Bacterial Universe.** Suppose the bacteria in the parable continued to double their population every minute. How long would it take until their volume exceeded the total volume of the observable universe, which is about 10^{79} m³? (*Hint*: Proceed by trial and error.)

FURTHER APPLICATIONS

29. **Human Doubling.** Human population in the year 2000 was about 6 billion. Suppose this population increases exponentially with a doubling time of 50 years.

    a. Extend the following table, showing the population at 50-year intervals under this scenario, until you reach the year 3000. Use scientific notation, as shown.

Year	Population
2000	6×10^9
2050	$12 \times 10^9 = 1.2 \times 10^{10}$
2100	$24 \times 10^9 = 2.4 \times 10^{10}$
⋮	⋮

    b. The total surface area of Earth is about 5.1×10^{14} m². Assuming that people could occupy all this area (in reality, most of it is ocean), approximately when would people be so crowded that every person would have only 1 m² of space?

    c. Suppose that, when we take into account the area needed to grow food and to find other resources, each person actually requires about 10^4 m² of area to survive. About when would we reach this limit?

d. Suppose that we learn to colonize other planets and moons in our solar system. The total surface area of the worlds in our solar system that could potentially be colonized (not counting gas planets such as Jupiter) is roughly five times the surface area of Earth. Under the assumptions of part c, could humanity fit in our solar system in the year 3000? Explain.

30. **Doubling Time vs. Initial Amount.**

a. Would you rather start with one penny ($0.01) and double your wealth every day or start with one dime ($0.10) and double your wealth every five days (assuming you want to get rich)? Explain.

b. Would you rather start with one penny ($0.01) and double your wealth every day or start with $1000 and double your wealth every two days (assuming you want to get rich in the long run)? Explain.

c. Which is more important in determining how fast exponential growth occurs: the doubling time or the initial amount? Explain.

WEB PROJECTS

31. **Computing Power.** Choose an aspect of computing power (such as processor speed or memory chip capacity) and investigate its growth. Has the growth been exponential? How much longer is the exponential growth likely to continue? Explain.

32. **Web Growth.** Investigate the growth of the Web itself, in terms of both number of users and number of Web pages. Has the growth been linear or exponential? How do you think the growth will change in the future? Explain.

IN YOUR WORLD

33. **Linear Growth.** Identify at least two news stories that describe a quantity undergoing linear growth or decay. Describe the growth or decay process.

34. **Exponential Growth.** Identify at least two news stories that describe a quantity undergoing exponential growth or decay. Describe the growth or decay process.

UNIT 8B Doubling Time and Half-Life

Exponential growth leads to repeated doublings and exponential decay leads to repeated *halvings.* However, in most cases of exponential growth or decay, we are given the rate of growth or decay—usually as a percentage—rather than the time required for doubling or halving. In this unit, we'll convert between growth (or decay) rates and doubling (or halving) times.

Doubling Time

The time required for each doubling in exponential growth is called the **doubling time**. For example, the doubling time for the magic penny (see Unit 8A) was one day, because your wealth doubled each day. The doubling time for the bacteria in the bottle was one minute.

Given the doubling time, we can easily calculate the value of a quantity at any time. Consider an initial population of 10,000 that grows with a doubling time of 10 years:

- In 10 years, or one doubling time, the population increases by a factor of 2, to a new population of $2 \times 10{,}000 = 20{,}000$.
- In 20 years, or two doubling times, the population increases by a factor of $2^2 = 4$, to a new population of $4 \times 10{,}000 = 40{,}000$.
- In 30 years, or three doubling times, the population increases by a factor of $2^3 = 8$, to a new population of $8 \times 10{,}000 = 80{,}000$.

We can generalize this idea by letting t be the amount of time that has passed and T_{double} be the doubling time. Note that after $t = 30$ years with a doubling time of $T_{\text{double}} = 10$ years, there have been $t/T_{\text{double}} = 30/10 = 3$ doublings. Generalizing, we find that the number of doublings after a time t is t/T_{double}. That is, the size of the population after time t is the initial population times $2^{t/T_{\text{double}}}$.

CALCULATIONS WITH THE DOUBLING TIME

After a time t, an exponentially growing quantity with a doubling time of T_{double} increases in size by a factor of $2^{t/T_{\text{double}}}$. The new value of the growing quantity is related to its initial value (at $t = 0$) by

$$\text{new value} = \text{initial value} \times 2^{t/T_{\text{double}}}$$

Time Out to Think

Consider an initial population of 10,000 that grows with a doubling time of 10 years. Confirm that the above formula gives a population of 80,000 after 30 years, as we found earlier. What does the formula predict for the population after 50 years?

Example 1 Doubling with Compound Interest

Compound interest (Unit 4B) produces exponential growth because an interest-bearing account grows by the same percentage each year. Suppose your bank account has a doubling time of 13 years. By what factor does your balance increase in 50 years?

Solution The doubling time is $T_{\text{double}} = 13$ years, so after $t = 50$ years your balance increases by a factor of

$$2^{t/T_{\text{double}}} = 2^{50\,\text{yr}/13\,\text{yr}} = 2^{3.8462} \approx 14.382$$

For example, if you start with a balance of \$1000, in 50 years it will grow to \$1000 × 14.382 = \$14,382.

Now try Exercises 25–32.

Example 2 World Population Growth

World population doubled from 3 billion in 1960 to 6 billion in 2000. Suppose that world population continued to grow (from 2000 on) with a doubling time of 40 years. What would the population be in 2030? in 2200?

Solution The doubling time is $T_{\text{double}} = 40$ years. If we let $t = 0$ represent 2000, the year 2030 is $t = 30$ years later. If the 2000 population of 6 billion is used as the initial value, the population in 2030 would be

$$\begin{aligned}\text{new value} &= \text{initial value} \times 2^{t/T_{\text{double}}}\\ &= 6\text{ billion} \times 2^{30\,\text{yr}/40\,\text{yr}}\\ &= 6\text{ billion} \times 2^{0.75} \approx 10.1\text{ billion}\end{aligned}$$

By 2200, which is $t = 200$ years after 2000, the population would reach

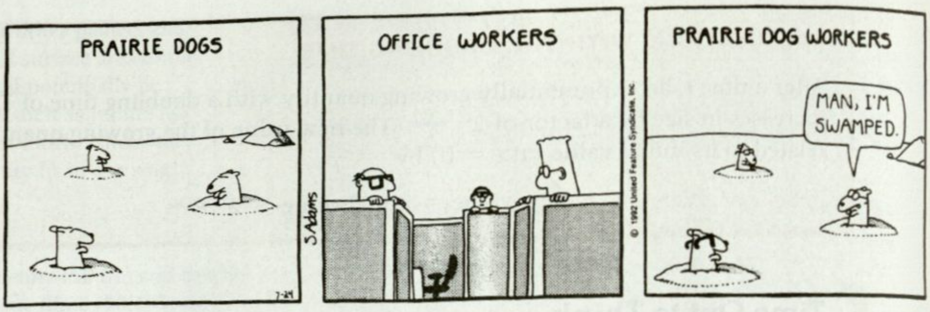
$$\begin{aligned}\text{new value} &= \text{initial value} \times 2^{t/T_{\text{double}}}\\ &= 6\text{ billion} \times 2^{200\,\text{yr}/40\,\text{yr}}\\ &= 6\text{ billion} \times 2^{5} = 192\text{ billion}\end{aligned}$$

If world population continued to grow at the same rate it did between 1960 and 2000, it would reach 10 billion by 2030 and 192 billion by 2200.

Now try Exercises 33–34.

Time Out to Think

Do you think that it's really possible for the human population on Earth to reach 192 billion? Why or why not?



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The Approximate Doubling Time Formula

Consider an ecological study of a prairie dog community. The community contains 100 prairie dogs when the study begins, and researchers soon determine that the population is increasing at a rate of 10% per month. That is, each month the population grows to 110% of, or 1.1 times, its previous value (see the "*of* versus *more than*" rule in Unit 3A). Table 8.3 tracks the population growth (rounded to the nearest whole number).

TABLE 8.3 Growth of a Prairie Dog Community

Month	Population	Month	Population
0	100	8	$(1.1)^8 \times 100 = 214$
1	$(1.1)^1 \times 100 = 110$	9	$(1.1)^9 \times 100 = 236$
2	$(1.1)^2 \times 100 = 121$	10	$(1.1)^{10} \times 100 = 259$
3	$(1.1)^3 \times 100 = 133$	11	$(1.1)^{11} \times 100 = 285$
4	$(1.1)^4 \times 100 = 146$	12	$(1.1)^{12} \times 100 = 314$
5	$(1.1)^5 \times 100 = 161$	13	$(1.1)^{13} \times 100 = 345$
6	$(1.1)^6 \times 100 = 177$	14	$(1.1)^{14} \times 100 = 380$
7	$(1.1)^7 \times 100 = 195$	15	$(1.1)^{15} \times 100 = 418$

Note that the population nearly doubles (to 195) after 7 months, then nearly doubles again (to 380) after 14 months. This roughly seven-month doubling time is related to the 10% growth rate as follows:

$$\text{doubling time} \approx \frac{70}{\text{percentage growth rate}} = \frac{70}{10/\text{mo}} = 7 \text{ mo}$$

This formula, in which the doubling time is approximately 70 divided by the percentage growth rate, works whenever the growth rate is relatively small (less than about 15%). It is often called the rule of 70.

APPROXIMATE DOUBLING TIME FORMULA (RULE OF 70)

For a quantity growing exponentially at a rate of P% per time period, the doubling time is *approximately*

$$T_{\text{double}} \approx \frac{70}{P}$$

This approximation works best for small growth rates and breaks down for growth rates over about 15%.

Example 3 Population Doubling Time

World population was about 6.0 billion in 2000 and was growing at a rate of about 1.4% per year. What is the approximate doubling time at this growth rate? If this growth rate were to continue, what would world population be in 2030? Compare to the result in Example 2.

Solution We can use the approximate doubling time formula because the growth rate is much less than 15%. The percentage growth rate of 1.4% per year means we set $P = 1.4/\text{yr}$:

$$T_{\text{double}} \approx \frac{70}{P} = \frac{70}{1.4/\text{yr}} = 50 \text{ yr}$$

The doubling time is approximately 50 years. The year 2030 is $t = 30$ years after 2000, so the population in 2030 would be

$$\begin{aligned} \text{new value} &= \text{initial value} \times 2^{t/T_{\text{double}}} \\ &= 6.0 \text{ billion} \times 2^{30\,\text{yr}/50\,\text{yr}} \\ &= 6.0 \text{ billion} \times 2^{0.6} \approx 9.1 \text{ billion} \end{aligned}$$

At a 1.4 % growth rate, world population would be about 9 billion in 2030. This is a billion fewer people than predicted in Example 2, in which the doubling time was assumed to be 40 years rather than 50 years. **Now try Exercises 35–36.**

Example 4 Solving the Doubling Time Formula

World population doubled in the 40 years from 1960 to 2000. What was the average percentage growth rate during this period? Contrast this growth rate with the 2000 growth rate of 1.4% per year.

Solution We can answer the question by solving the approximate doubling time formula for P. We multiply both sides of the formula by P and divide both sides by T_{double}. You should confirm that this gives the following result:

$$P \approx \frac{70}{T_{\text{double}}}$$

Substituting $T_{\text{double}} = 40$ years, we find

$$P \approx \frac{70}{T_{\text{double}}} = \frac{70}{40 \text{ yr}} = 1.75/\text{yr}$$

The average population growth rate between 1960 and 2000 was about $P\% = 1.75\%$ per year. This is 0.35 percentage point higher than the 2000 growth rate of 1.4% per year. **Now try Exercises 37–40.**

Exponential Decay and Half-Life

Exponential *decay* occurs whenever a quantity *decreases* by the same percentage in every fixed time period (for example, by 20% every year). In that case, the value of the quantity repeatedly decreases to half its value, with each halving occurring in a time called the **half-life**.

You may have heard half-lives described for radioactive materials such as uranium or plutonium. For example, radioactive plutonium-239 (Pu-239) has a half-life of about 24,000 years. To understand the meaning of the half-life, suppose that 100 pounds of Pu-239 is deposited at a nuclear waste site. The plutonium gradually decays into other substances as follows:

- In 24,000 years, or one half-life, the amount of Pu-239 declines to 1/2 its original value, or to $(1/2) \times 100$ pounds $= 50$ pounds.

BY THE WAY

Plutonium-239 is the chemical element plutonium in a form (or *isotope*) with atomic weight 239. Atomic weight is the total number of protons and neutrons in the nucleus. Because all plutonium nuclei have 94 protons, Pu-239 nuclei have $239 - 94 = 145$ neutrons.

Historical Note

The atomic bomb that devastated Nagasaki during World War II generated its destructive power by fission of plutonium-239. (The Hiroshima bomb used uranium-235.)

- In 48,000 years, or two half-lives, the amount of Pu-239 declines to $(1/2)^2 = 1/4$ its original value, or to $(1/4) \times 100$ pounds $= 25$ pounds.
- In 72,000 years, or three half-lives, the amount of Pu-239 declines to $(1/2)^3 = 1/8$ its original value, or to $(1/8) \times 100$ pounds $= 12.5$ pounds.

We can generalize this idea much as we did earlier for the doubling time. A single halving reduces a quantity by a factor of $1/2$, two halvings reduce it by a factor of $(1/2)^2$, and three halvings reduce it by a factor of $(1/2)^3$. If we let t be the amount of time that has passed and T_{half} be the half-life, then the number of halvings after a time t is t/T_{half}. That is, the quantity after time t is the original quantity times this factor of $(1/2)^{t/T_{\text{half}}}$.

CALCULATIONS WITH THE HALF-LIFE

After a time t, an exponentially decaying quantity with a half-life of T_{half} decreases in size by a factor of $(1/2)^{t/T_{\text{half}}}$. The new value of the decaying quantity is related to its initial value (at $t = 0$) by

$$\text{new value} = \text{initial value} \times \left(\frac{1}{2}\right)^{t/T_{\text{half}}}$$

BY THE WAY

Ordinary carbon is carbon-12, which is stable (not radioactive). Carbon-14 is produced in the Earth's atmosphere by high-energy particles coming from the Sun. It mixes with ordinary carbon and therefore becomes incorporated into living tissue through respiration (breathing).

Example 5 Carbon-14 Decay

Radioactive carbon-14 has a half-life of about 5700 years. It collects in organisms only while they are alive. Once they are dead, it only decays. What fraction of the carbon-14 in an animal bone still remains 1000 years after the animal has died?

Solution The half-life is $T_{\text{half}} = 5700$ years, so the fraction of the initial amount remaining after $t = 1000$ years is

$$\left(\frac{1}{2}\right)^{t/T_{\text{half}}} = \left(\frac{1}{2}\right)^{1000\,\text{yr}/5700\,\text{yr}} \approx 0.885$$

For example, if the bone originally contained 1 kilogram of carbon-14, the amount remaining after 1000 years is approximately 0.885 kilogram. We can use this idea to determine the age of bones found at archaeological sites, as we'll discuss in Unit 9C.

Now try Exercises 41–44.

Example 6 Plutonium After 100,000 Years

Suppose that 100 pounds of Pu-239 is deposited at a nuclear waste site. How much of it will still be present in 100,000 years?

Solution The half-life of Pu-239 is $T_{\text{half}} = 24{,}000$ years. Given an initial amount of 100 pounds, the amount remaining after $t = 100{,}000$ years is

$$\text{new value} = \text{initial value} \times \left(\frac{1}{2}\right)^{t/T_{\text{half}}} = 100\text{ lb} \times \left(\frac{1}{2}\right)^{100{,}000\,\text{yr}/24{,}000\,\text{yr}} \approx 5.6\text{ lb}$$

About 5.6 pounds of the original 100 pounds of Pu-239 will still be present in 100,000 years.

Now try Exercises 45–48.

Time Out to Think

Plutonium, which is not found naturally on the Earth, is made in nuclear reactors for use both as fuel for nuclear power plants and for nuclear weapons. Based on its half-life, explain why the safe disposal of Pu-239 poses a significant challenge.

The Approximate Half-Life Formula

The approximate doubling time formula (the rule of 70) found earlier works equally well for exponential decay if we replace the doubling time with the half-life and the percentage growth rate with the percentage decay rate.

APPROXIMATE HALF-LIFE FORMULA

For a quantity decaying exponentially at a rate of $P\%$ per time period, the half-life is *approximately*

$$T_{\text{half}} \approx \frac{70}{P}$$

This approximation works best for small decay rates and breaks down for decay rates over about 15%.

Take Note

Some books treat P as negative for exponential decay, in which case the half-life is defined as $70/|P|$, where $|P|$ is the absolute value of P.

Example 7 Devaluation of Currency

Suppose that inflation causes the value of the Russian ruble to fall at a rate of 12% per year (relative to the dollar). At this rate, approximately how long does it take for the ruble to lose half its value?

Solution We can use the approximate half-life formula because the decay rate is less than 15%. The 12% decay rate means we set $P = 12/\text{yr}$:

$$T_{\text{half}} \approx \frac{70}{P} = \frac{70}{12/\text{yr}} \approx 5.8 \text{ yr}$$

The half-life is a little less than six years, meaning that the ruble loses half its value (against the dollar) in six years.

Now try Exercises 49–52.

Exact Formulas for Doubling Time and Half-Life

The approximate doubling time and half-life formulas are useful because they are easy to remember. However, for more precise work or for cases of larger rates where the approximate formulas break down, we need the exact formulas, given below. In Unit 9C, we will see how they are derived. These formulas use the **fractional growth rate**, defined as $r = P/100$, with r positive for growth and negative for decay. For example, if the percentage growth rate is 5% per year, the fractional growth rate is $r = 0.05$ per year. For a 5% *decay* rate per year, the fractional growth rate is $r = -0.05$ per year.

EXACT DOUBLING TIME AND HALF-LIFE FORMULAS

For an exponentially growing quantity with a fractional growth rate r, the doubling time is

$$T_{\text{double}} = \frac{\log_{10} 2}{\log_{10}(1 + r)}$$

For an exponentially decaying quantity, we use a *negative* value for r (for example, if the decay rate is $P = 5\%$ per year, we set $r = -0.05$ per year). The half-life is

$$T_{\text{half}} = -\frac{\log_{10} 2}{\log_{10}(1 + r)}$$

Note that *the units of time used for T and r must be the same.* For example, if the fractional growth rate is 0.05 per *month*, then the doubling time will also be measured in months. Also note that the formulas ensure that both T_{double} and T_{half} have positive values.

Example 8 Large Growth Rate

A population of rats is growing at a rate of 80% per month. Find the exact doubling time for this growth rate and compare it to the doubling time found with the approximate doubling time formula.

Solution The growth rate of 80% per month means $P = 80/\text{mo}$ or $r = 0.8/\text{mo}$. The doubling time is

$$T_{\text{double}} = \frac{\log_{10} 2}{\log_{10}(1 + 0.8)} = \frac{0.301\,030}{\log_{10}(1.8)} = \frac{0.301\,030}{0.255\,273} \approx 1.18 \text{ mo}$$

A Brief Review Logarithms

A **logarithm** (or **log**, for short) is a power or exponent. In this book, we will focus on base 10 logs, also called **common logs**, which are defined as follows:

$\log_{10} x$ is the power to which 10 must be raised to obtain x.

You may find it easier to remember the meaning with a less technical definition:

$\log_{10} x$ means "10 to what power equals x?"

For example:

$$\begin{aligned}
\log_{10} 1000 &= 3 && \text{because } 10^3 = 1000\\
\log_{10} 10{,}000{,}000 &= 7 && \text{because } 10^7 = 10{,}000{,}000\\
\log_{10} 1 &= 0 && \text{because } 10^0 = 1\\
\log_{10} 0.1 &= -1 && \text{because } 10^{-1} = 0.1\\
\log_{10} 30 &\approx 1.477 && \text{because } 10^{1.477} \approx 30
\end{aligned}$$

Four important rules follow directly from the definition of a logarithm.

1. Taking the logarithm of a power of 10 gives the power. That is,
$$\log_{10} 10^x = x$$
2. Raising 10 to a power that is the logarithm of a number gives back the number. That is,
$$10^{\log_{10} x} = x \qquad (x > 0)$$
3. Because powers of 10 are multiplied by adding their exponents, we have the *addition rule* for logarithms:
$$\log_{10} xy = \log_{10} x + \log_{10} y \qquad (x > 0 \text{ and } y > 0)$$
4. We can "bring down" an exponent within a logarithm by applying the *power rule* for logarithms:
$$\log_{10} a^x = x \times \log_{10} a \qquad (a > 0)$$

Most calculators have a key to compute $\log_{10}$ of any positive number. You should find this key on your calculator and use it to verify that $\log_{10} 1000 = 3$ and $\log_{10} 2 \approx 0.301\,030$.

Example: Given that $\log_{10} 2 \approx 0.301\,030$, find each of the following:

a. $\log_{10} 8$

b. $10^{\log_{10} 2}$

c. $\log_{10} 200$

Solution:

a. We notice that $8 = 2^3$. Therefore, from Rule 4,
$$\begin{aligned}\log_{10} 8 &= \log_{10} 2^3\\ &= 3 \times \log_{10} 2 \approx 3 \times 0.301\,030 = 0.903\,09\end{aligned}$$

b. From Rule 2,
$$10^{\log_{10} 2} = 2$$

c. We notice that $200 = 2 \times 100 = 2 \times 10^2$. Therefore, from Rule 3,
$$\log_{10} 200 = \log_{10}(2 \times 10^2) = \log_{10} 2 + \log_{10} 10^2$$

From Rule 1, we know that $\log_{10} 10^2 = 2$, so
$$\begin{aligned}\log_{10} 200 &= \log_{10}(2 \times 10^2) = \log_{10} 2 + \log_{10} 10^2\\ &\approx 0.301\,030 + 2 = 2.301\,030\end{aligned}$$

Example: Someone tells you that $\log_{10} 600 = 5.778$. Should you believe it?

Solution: Because 600 is between 100 and 1000, $\log_{10} 600$ must be between $\log_{10} 100$ and $\log_{10} 1000$. From Rule 1, we find that $\log_{10} 100 = \log_{10} 10^2 = 2$ and $\log_{10} 1000 = \log_{10} 10^3 = 3$. Therefore, $\log_{10} 600$ must be between 2 and 3, so the claimed answer of 5.778 must be wrong.

Now try Exercises 13–24.

The doubling time is almost 1.2 months. Note that this answer makes sense: With the population growing by 80% in a month, we expect it to take a little over a month to grow by 100% (which is a doubling). In contrast, the approximate doubling time formula predicts a doubling time of $70/P = 70/80 = 0.875$ month, which is less than one month. We see that the approximate formula does not work well for large growth rates.

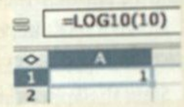
Now try Exercises 53–54.

USING TECHNOLOGY

Logarithms

Most calculators have a key for computing common (base 10) logarithms, usually labeled *log*. Note, however, that some calculators use log to mean the natural (base *e*) logarithm; you can check by testing that your calculator correctly returns $\log_{10} 10 = 1$.

In Excel, use the built-in function LOG10, as shown in the screen shot below. (You can also use the LOG function, which can be used for any base but uses base 10 by default.)



Example 10 Ruble Revisited

Suppose the Russian ruble is falling in value against the dollar at 12% per year. Using the exact half-life formula, determine how long it takes the ruble to lose half its value. Compare your answer to the approximate answer found in Example 7.

Solution The percentage decay rate is $P = 12\%/\text{yr}$. Because this is a rate of *decay*, we set the fractional growth rate to $r = -0.12/\text{yr}$. The half-life is

$$T_{\text{half}} = -\frac{\log_{10} 2}{\log_{10}(1 - 0.12)} = \frac{0.301\,030}{-0.055\,517} \approx 5.42 \text{ yr}$$

The ruble loses half its value against the dollar in about 5.42 years. This result is about 0.4 year less than the 5.8 years obtained with the approximate formula. We see that the approximate formula is reasonably accurate for the 12% rate.

Now try Exercises 55–56.

QUICK QUIZ

Choose the best answer to each of the following questions. Explain your reasoning with one or more complete sentences.

1. Suppose an investment rises consistently so that its value doubles every 7 years. By what factor will its value rise in 30 years?

   a. $2^{30/7}$ b. $2^{30} - 2^{7}$ c. 7×2^{30}

2. Suppose your salary is increasing at a rate of 2.5% per year. Then your salary will double in approximately

   a. 2.5 years. b. $\frac{70}{2.5}$ years. c. $\frac{2.5}{70}$ years.

3. Which of the following is *not* a good approximation of a doubling time?

   a. Inflation running at 35% per year will cause prices to double in about 2 years.

   b. A town growing at 2% per year will double its population in about 35 years.

   c. A bank account balance growing at 7% per year will double in about 10 years.

4. A town's population doubles in 15 years. Its percentage growth rate is approximately

   a. 15% per year. b. $\frac{70}{15}$% per year. c. $\frac{15}{70}$% per year.

5. Radioactive tritium (hydrogen-3) has a half-life of about 12 years, which means that if you start with 1 kg of tritium, 0.5 kg will decay during the first 12 years. How much will decay during the next 12 years?

   a. $\frac{12}{0.5}$ kg b. 0.5 kg c. 0.25 kg

6. Radioactive uranium-235 has a half-life of about 700 million years. Suppose a rock is 2.8 billion years old. What fraction of the rock's original uranium-235 still remains?

   a. $\frac{1}{2}$ b. $\frac{1}{16}$ c. $\frac{1}{700}$

7. The population of an endangered species is falling at a rate of 7% per year. Approximately how long will it take the population to drop by half?

   a. 7 years b. 10 years c. 17 years

8. $\log_{10} 10^8 =$

   a. 100,000,000 b. 108 c. 8

9. A rural population is falling at a rate of 20% per decade. If you wish to calculate its exact half-life, you should set the fractional growth rate per decade to

a. $r = 20$.

b. $r = 0.2$.

c. $r = -0.2$.

10. A new company's revenues are growing 15% per year. The doubling time for its revenues is

a. $\dfrac{\log_{10}2}{\log_{10}1.15}$ years.

b. $\dfrac{\log_{10}2}{\log_{10}0.85}$ years.

c. $\dfrac{\log_{10}(1 + 0.15)}{\log_{10}2}$ years.

Exercises 8B

REVIEW QUESTIONS

1. What is a doubling time? Suppose a population has a doubling time of 25 years. By what factor will it grow in 25 years? in 50 years? in 100 years?
2. Given a doubling time, explain how you calculate the value of an exponentially growing quantity at any time *t*.
3. State the approximate doubling time formula and the conditions under which it works well. Give an example.
4. What is a half-life? Suppose a radioactive substance has a half-life of 1000 years. What fraction will be left after 1000 years? after 2000 years? after 4000 years?
5. Given a half-life, explain how you calculate the value of an exponentially decaying quantity at any time *t*.
6. State the approximate half-life formula and the conditions under which it works well. Give an example.
7. Briefly describe exact doubling time and half-life formulas. Explain all their terms.
8. Give an example in which it is important to use the exact doubling time or half-life formula, rather than the approximate formula. Explain why the approximate formula does not work well in this case.

DOES IT MAKE SENSE?

Decide whether each of the following statements makes sense (or is clearly true) or does not make sense (or is clearly false). Explain your reasoning.

9. Our town is growing with a doubling time of 25 years, so its population will triple in 50 years.
10. Our town is growing at a rate of 7% per year, so it will double in population about every 10 years.
11. A toxic chemical decays with a half-life of 10 years, so half of it will be gone 10 years from now and all the rest will be gone 20 years from now.
12. The half-life of plutonium-239 is about 24,000 years, so we can expect some of the plutonium produced in recent decades to still be around 100,000 years from now.

BASIC SKILLS & CONCEPTS

13–24: Logarithms. Refer to the Brief Review on p. 486. Determine whether each statement is true or false *without* doing any calculations. Explain your reasoning.

13. $10^{0.928}$ is between 1 and 10.
14. $10^{3.334}$ is between 500 and 1000.
15. $10^{-5.2}$ is between $-100{,}000$ and $-1{,}000{,}000$.
16. $10^{-2.67}$ is between 0.001 and 0.01.
17. $\log_{10}\pi$ is between 3 and 4.
18. $\log_{10}96$ is between 3 and 4.
19. $\log_{10}1{,}600{,}000$ is between 16 and 17.
20. $\log_{10}(8 \times 10^9)$ is between 9 and 10.
21. $\log_{10}\left(\frac{1}{4}\right)$ is between -1 and 0.
22. $\log_{10}0.00045$ is between 5 and 6.
23. Using the approximation $\log_{10}2 = 0.301$, find each of the following *without* a calculator.

a. $\log_{10}8$ b. $\log_{10}2000$ c. $\log_{10}0.5$

d. $\log_{10}64$ e. $\log_{10}1/8$ f. $\log_{10}0.2$

24. Using the approximation $\log_{10}5 = 0.699$, find each of the following *without* a calculator.

a. $\log_{10}50$ b. $\log_{10}5000$ c. $\log_{10}0.05$

d. $\log_{10}25$ e. $\log_{10}0.20$ f. $\log_{10}0.04$

25–32: Doubling Time. Each exercise gives a doubling time for an exponentially growing quantity. Answer the questions that follow.

25. The doubling time of a population of fruit flies is 8 hours. By what factor does the population increase in 24 hours? in 1 week?
26. The doubling time of a bank account balance is 20 years. By what factor does it grow in 40 years? in 100 years?
27. The doubling time of a city's population is 17 years. How long does it take for the population to quadruple?
28. Prices are rising with a doubling time of 2 months. By what factor do prices increase in a year?
29. The initial population of a town is 10,000, and it grows with a doubling time of 10 years. What will the population be in 12 years? in 24 years?
30. The initial population of a town is 10,000, and it grows with a doubling time of 8 years. What will the population be in 12 years? in 24 years?
31. The number of cells in a tumor doubles every 2.5 months. If the tumor begins as a single cell, how many cells will there be after 2 years? after 4 years?
32. The number of cells in a tumor doubles every 6 months. If the tumor begins with a single cell, how many cells will there be after 6 years? after 10 years?

33–34: World Population. In 2009 world population was 6.8 billion. Use the given doubling time to predict the population in 2019, 2059, and 2109.

33. Assume a doubling time of 45 years.

34. Assume a doubling time of 60 years.

35. **Rabbits.** A community of rabbits begins with an initial population of 100 and grows 7% per month. Make a table, similar to Table 8.3, that shows the population for each of the next 15 months. Based on the table, find the doubling time of the population and briefly discuss how well the approximate doubling time formula works for this case.

36. **Mice.** A community of mice begins with an initial population of 1000 and grows 20% per month. Make a table, similar to Table 8.3, that shows the population for each of the next 15 months. Based on the table, find the doubling time of the population and briefly discuss how well the approximate doubling time formula works for this case.

37–40: Doubling Time Formula. Use the approximate doubling time formula (rule of 70). Discuss whether the formula is valid for the case described.

37. The Consumer Price Index is increasing at a rate of 4% per year. What is its doubling time? By what factor will prices increase in 3 years?

38. A city's population is growing at a rate of 3.5% per year. What is its doubling time? By what factor will the population increase in 50 years?

39. Prices are rising at a rate of 0.3% per month. What is their doubling time? By what factor will prices increase in 1 year? in 8 years?

40. Oil consumption is increasing at a rate of 2.2% per year. What is its doubling time? By what factor will oil consumption increase in a decade?

41–48: Half-Life. Each exercise gives a half-life for an exponentially decaying quantity. Answer the questions that follow.

41. The half-life of a radioactive substance is 50 years. If you start with some amount of this substance, what fraction will remain in 100 years? in 200 years?

42. The half-life of a radioactive substance is 400 years. If you start with some amount of this substance, what fraction will remain in 120 years? in 2500 years?

43. The half-life of a drug in the bloodstream is 18 hours. What fraction of the original drug dose remains in 24 hours? in 48 hours?

44. The half-life of a drug in the bloodstream is 4 hours. What fraction of the original drug dose remains in 24 hours? in 48 hours?

45. The current population of a threatened animal species is 1 million, but it is declining with a half-life of 20 years. How many animals will be left in 30 years? in 70 years?

46. The current population of a threatened animal species is 1 million, but it is declining with a half-life of 25 years. How many animals will be left in 20 years? in 40 years?

47. Cobalt-56 has a half-life of 77 days. If you start with 1 kilogram of cobalt-56, how much will remain after 150 days? after 300 days?

48. Radium-226 is a metal with a half-life of 1600 years. If you start with 1 kilogram of radium-226, how much will remain after 1000 years? after 10,000 years?

49–52: Half-Life Formula. Use the approximate half-life formula. Discuss whether the formula is valid for the case described.

49. Urban encroachment is causing the area of a forest to decline at a rate of 7% per year. What is the half-life of the forest? What fraction of the forest will remain in 30 years?

50. A clean-up project is reducing the concentration of a pollutant in the water supply, with a 3.5% decrease per week. What is the half-life of the concentration of the pollutant? What fraction of the original amount of the pollutant will remain when the project ends after 1 year (52 weeks)?

51. Poaching is causing a population of elephants to decline by 8% per year. What is the half-life for the population? If there are 10,000 elephants today, how many will remain in 50 years?

52. The production of a gold mine is declining by 6% per year. What is the half-life for the production decline? If its current annual production is 5000 kilograms, what will its production be in 15 years?

53–56: Exact Formulas. Compare the doubling times found with the approximate and exact doubling time formulas. Then use the exact doubling time formula to answer the given question.

53. Inflation is causing prices to rise at a rate of 12% per year. For an item that costs $500 today, what will the price be in 4 years?

54. Hyperinflation is driving up prices at a rate of 80% per month. For an item that costs $1000 today, what will the price be in 1 year?

55. A nation of 100 million people is growing at a rate of 4% per year. What will its population be in 30 years?

56. A family of 100 termites invades your house and grows at a rate of 20% per week. How many termites will be in your house after 1 year (52 weeks)?

FURTHER APPLICATIONS

57. **Plutonium on Earth.** Scientists believe that Earth once had naturally existing plutonium-239. Suppose Earth had 10 trillion tons of Pu-239 when it formed. Given plutonium's half-life of 24,000 years and Earth's current age of 4.6 billion years, how much would remain today? Use your answer to explain why plutonium is not found naturally on Earth today.

58. **Nuclear Weapons.** Thermonuclear weapons use tritium for their nuclear reactions. Tritium is a radioactive form of hydrogen (containing 1 proton and 2 neutrons) with a half-life of about 12 years. Suppose a nuclear weapon contains 1 kilogram of tritium. How much will remain in 50 years? Use your answer to explain why thermonuclear weapons require regular maintenance.

WEB PROJECTS

59. **National Growth Rates.** Find growth rates, doubling times, and population projections tabulated for different countries. Select several countries from several continents and record relevant growth data. Comment on whether the doubling times and growth rates are consistent. Discuss how these data are used to make population projections.

60. **World Population Growth.** The Web site for the U.S. Census Bureau contains a wealth of data on world population. Visit this site and gather data on world population growth over the last 50 years. Estimate the population growth rate over each decade. Compute the associated doubling times. Write a two-paragraph statement on the trends that you observe.

IN YOUR WORLD

61. **Doubling Time.** Find a news story that gives an exponential growth rate. Find the approximate doubling time from the growth rate and discuss the implications of the growth.

62. **Radioactive Half-Life.** Find a news story that discusses some type of radioactive material. If it is not given, look up the half-life of the material. Discuss the implications for disposal of the material.

TECHNOLOGY EXERCISES

63. **Logarithms I.** Use a calculator or Excel to find each of the logarithms in Exercise 23. Give answers to 6 decimal places.

64. **Logarithms II.** Use a calculator or Excel to find each of the logarithms in Exercise 24. Give answers to 6 decimal places.

UNIT 8C Real Population Growth

Perhaps the most important application of exponential growth concerns human population. From the time of the earliest humans more than 2 million years ago until about 10,000 years ago, human population probably never exceeded 10 million. The advent of agriculture brought about more rapid population growth. Human population reached 250 million by C.E. 1 and continued growing slowly to about 500 million by 1650.

Exponential growth set in with the Industrial Revolution. Our rapidly developing ability to grow food and exploit natural resources allowed us to build more homes for more people. Meanwhile, improvements in medicine and health science lowered death rates dramatically. In fact, world population began growing at a rate exceeding that of steady exponential growth, in which the doubling time would have remained constant. Population doubled from 500 million to 1 billion in the 150 years from 1650 to 1800. It then doubled again, to 2 billion, by 1922, only about 120 years. The next doubling, to 4 billion, was complete by 1974, a doubling time of only 52 years. World population is estimated to have reached 6 billion during 1999, and 6.8 billion in 2009. Figure 8.3 shows the estimated human population over the past 12,000 years.

To put current world population growth in perspective, consider the following facts:

- Every four years, the world adds as many people as the total population of the United States.

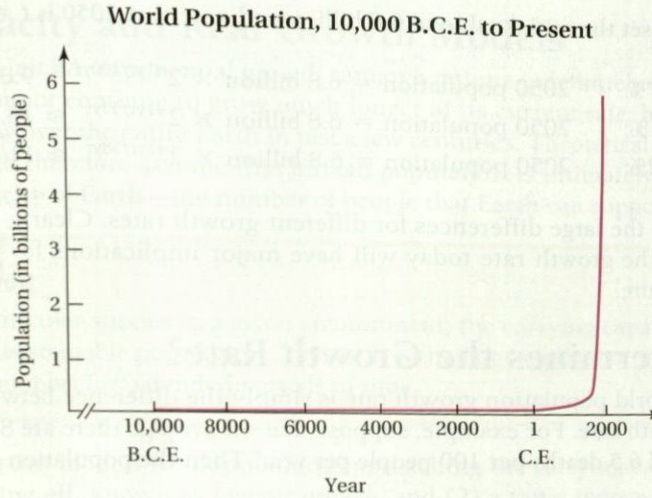


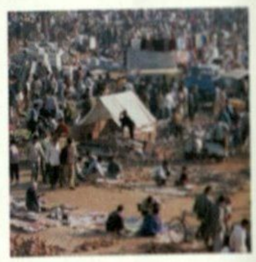
FIGURE 8.3 World population.

- Each month, world population increases by the equivalent of the population of Switzerland.
- While you study during the next hour and a half, world population will increase by about 10,000 people.

Projections of future population growth have large uncertainties. Nevertheless, current trends suggest that world population will reach 7 billion by about 2012 and 8 billion by 2025. Even in the United States, where population is growing more slowly than the world average, the population is expected to increase by *100 million* people within the next 50 years. Fortunately, while the absolute increase in population remains huge, these numbers indicate that the growth rate is declining. This fact makes many researchers suspect that the growth rate will continue its downward trend, thereby preventing a population catastrophe.

BY THE WAY

Most of the projected population growth (9 out of 10 people) will take place in the developing regions. By 2030, more than 1 billion people are expected to be added to Asia alone, and India is expected to overtake China as the most populous nation.



Example 1 Varying Growth Rate

The average annual growth rate for world population since 1650 has been about 0.7%. However, the annual rate has varied significantly. It peaked at about 2.1% during the 1960s and is currently (as of 2009) about 1.2%. Find the approximate doubling time for each of these growth rates. Use each to predict world population in 2050, based on a 2009 population of 6.8 billion.

Solution Using the approximate doubling time formula (Unit 8B), we find the doubling times for the three rates:

$$\textit{For 0.7\%:}\quad T_{\text{double}} \approx \frac{70}{P} = \frac{70}{0.7/\text{yr}} = 100 \text{ yr}$$

$$\textit{For 2.1\%:}\quad T_{\text{double}} \approx \frac{70}{P} = \frac{70}{2.1/\text{yr}} = 33 \text{ yr}$$

$$\textit{For 1.2\%:}\quad T_{\text{double}} \approx \frac{70}{P} = \frac{70}{1.2/\text{yr}} = 58 \text{ yr}$$

To predict world population in 2050, we use the formula

$$\text{new value} = \text{initial value} \times 2^{t/T_{\text{double}}}$$

We set the initial value to 6.8 billion and note that 2050 is $t = 41$ years after 2009:

For 0.7%: 2050 population $= 6.8 \text{ billion} \times 2^{41\text{yr}/100\text{yr}} \approx 9.0$ billion

For 2.1%: 2050 population $= 6.8 \text{ billion} \times 2^{41\text{yr}/33\text{yr}} \approx 16.1$ billion

For 1.2%: 2050 population $= 6.8 \text{ billion} \times 2^{41\text{yr}/58\text{yr}} \approx 11.1$ billion

Notice the large differences for different growth rates. Clearly, decisions we make to affect the growth rate today will have major implications for human population in the future. **Now try Exercises 13–16.**

BY THE WAY

For individual countries, the growth rate depends on immigration and emigration as well as on birth and death rates. In the United States, immigration comprises about half the overall growth rate. Of course, immigration and emigration do not affect *world* population, since no one is moving to or from the Earth.

What Determines the Growth Rate?

The world population growth rate is simply the difference between the birth rate and the death rate. For example, suppose that on average there are 8.5 births per 100 people and 6.5 deaths per 100 people per year. Then the population growth rate is

$$\frac{8.5}{100} - \frac{6.5}{100} = \frac{2}{100} = 0.02 = 2\%$$

OVERALL GROWTH RATE

The world population growth rate is the difference between the birth rate and the death rate:

$$\text{growth rate} = \text{birth rate} - \text{death rate}$$

Interestingly, birth rates dropped rapidly throughout the world during the past 60 years—the same period that has seen the largest population growth in history. Indeed, worldwide birth rates have never been lower than they are today. Today's rapid population growth comes from the fact that death rates have fallen even more dramatically.

Example 2 Birth and Death Rates

BY THE WAY

By 2009, the world birth rate had fallen to about 2.0 per 100 people and the death rate had fallen to about 0.8 per 100 people.

In 1950, the world birth rate was 3.7 births per 100 people and the world death rate was 2.0 deaths per 100 people. By 1975, the birth rate had fallen to 2.8 births per 100 people and the death rate to 1.1 deaths per 100 people. Contrast the growth rates in 1950 and 1975.

Solution In 1950, the overall growth rate was

$$\frac{3.7}{100} - \frac{2.0}{100} = \frac{1.7}{100} = 1.7\%$$

In 1975, the overall growth rate was

$$\frac{2.8}{100} - \frac{1.1}{100} = \frac{1.7}{100} = 1.7\%$$

Despite a dramatic fall in birth rates during the 25-year period, the growth rate remained unchanged because death rates fell equally dramatically. **Now try Exercises 17–20.**

Time Out to Think

Suppose that medical science finds a way to extend human lifespans significantly. How would this affect the population growth rate?

Carrying Capacity and Real Growth Models

As we saw in Unit 8A, exponential growth cannot continue indefinitely. Indeed, human population cannot continue to grow much longer at its current rate, because we'd be elbow to elbow over the entire Earth in just a few centuries. Theoretical models of population growth therefore assume that human population is ultimately limited by the carrying capacity of Earth—the number of people that Earth can support.

DEFINITION

For any particular species in a given environment, the **carrying capacity** is the maximum sustainable population. That is, it is the largest population the environment can support for extended periods of time.

Two important models for populations approaching the carrying capacity are (1) a gradual leveling off, known as *logistic growth,* and (2) a rapid increase followed by a rapid decrease, known as *overshoot and collapse.* Let's investigate each model.

Logistic Growth

A **logistic growth** model assumes that population growth gradually slows as the population approaches the carrying capacity. For example, if the carrying capacity is 12 billion people, a logistic model assumes that population growth will slow as this number is approached. The growth rate falls to zero as the carrying capacity is approached, allowing the population to remain steady at that level thereafter.

LOGISTIC GROWTH

When the population is small relative to the carrying capacity, **logistic growth** is exponential with a fractional growth rate close to the base growth rate r. As the population approaches the carrying capacity, the logistic growth rate approaches zero. The logistic growth rate at any particular time depends on the population at that time, the carrying capacity, and the base growth rate r:

$$\text{logistic growth rate} = r \times \left(1 - \frac{\text{population}}{\text{carrying capacity}}\right)$$

Figure 8.4 contrasts logistic and exponential growth for the same base growth rate r. In the exponential case, the growth rate stays equal to r at all times. In the logistic case, the growth rate starts out equal to r, so the logistic curve and the exponential curve

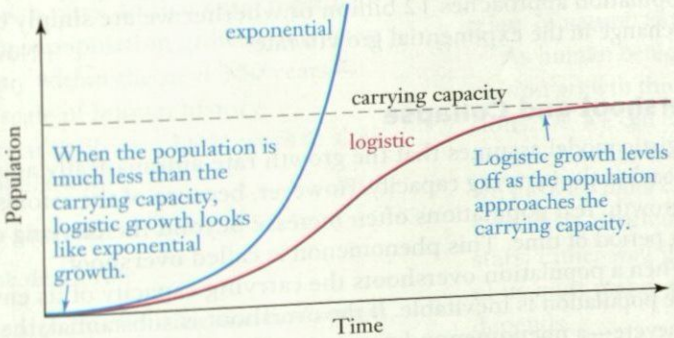


FIGURE 8.4 This graph contrasts exponential growth with logistic growth for the same base growth rate r.

look the same at early times. As time progresses, the logistic growth rate becomes ever smaller than r, and it finally reaches zero as the population levels out at the carrying capacity.

Example 3 Are We Growing Logistically?

Assume that the Earth's carrying capacity is 12 billion people. Given that the population growth rate peaked in the 1960s at about 2.1%, when the population was about 3 billion, is it reasonable to assume that human population has been following a logistic growth pattern since the 1960s? Is it reasonable to assume that population has been growing logistically throughout the past century? Explain.

Solution We need to compare the 2009 growth rate of about 1.2% (see Example 1) to the growth rate predicted by a logistic model. First, we must use the 1960s data to find the base growth rate r in the logistic model. You should confirm that solving the logistic growth rate formula for r gives

$$r = \frac{\text{growth rate}}{\left(1 - \dfrac{\text{population}}{\text{carrying capacity}}\right)}$$

Substituting the 1960s growth rate of 2.1% = 0.021 and population of 3 billion, along with a carrying capacity of 12 billion, we find

$$r = \frac{0.021}{\left(1 - \dfrac{3\text{ billion}}{12\text{ billion}}\right)} = \frac{0.021}{(1 - 0.25)} \approx 0.028$$

Now, we use this value of r to predict the growth rate for the 2009 population of about 6.8 billion:

$$\text{growth rate} = 0.028 \times \left(1 - \frac{6.8\text{ billion}}{12\text{ billion}}\right) \approx 0.012$$

This logistic model predicts a current growth rate of 1.2%, matching the actual current growth rate of 1.2%. Therefore, it is reasonable to say that human population has been growing logistically since the 1960s. However, human population has not been following logistic growth over longer periods. Logistic growth requires a continually decreasing growth rate, which is inconsistent with the growth rate peaking in the 1960s. In conclusion, it is still too early to know whether the growth rate will continue to decline as population approaches 12 billion or whether we are simply experiencing a temporary change in the exponential growth rate. **Now try Exercises 21–22.**

BY THE WAY

Overshoot and collapse characterizes many predator-prey populations. The population of a predator increases rapidly, causing the prey population to collapse. Once the prey population collapses, the predator population must also collapse because of lack of food. Once the predator population collapses, the prey population can begin to recover—as long as it has not gone extinct.

Overshoot and Collapse

A logistic model assumes that the growth rate automatically adjusts as the population approaches the carrying capacity. However, because of the astonishing rate of exponential growth, real populations often increase beyond the carrying capacity in a relatively short period of time. This phenomenon is called **overshoot**.

When a population overshoots the carrying capacity of its environment, a decrease in the population is inevitable. If the overshoot is substantial, the decrease can be rapid and severe—a phenomenon known as **collapse**. Figure 8.5 contrasts a logistic growth model with overshoot and collapse.

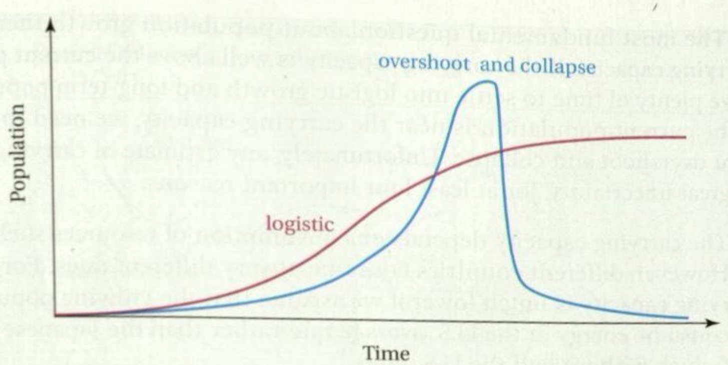


FIGURE 8.5 This graph contrasts logistic growth with overshoot and collapse.

Time Out to Think

The concept of carrying capacity can be applied to any localized environment. Consider the decline of past civilizations such as the ancient Greeks, Romans, Mayans, and Anasazi. Does an overshoot and collapse model describe the fall of any of these or other civilizations? Explain.

What Is the Carrying Capacity?

Given that human population cannot grow exponentially forever, logistic growth is clearly preferable to any kind of overshoot and collapse. Logistic growth means a sustainable future population, while overshoot and collapse might mean the end of our civilization.

PRACTICAL MATTERS

Choosing Our Fate

As the parable of the bacteria in a bottle (Unit 8A) showed, exponential growth cannot continue indefinitely. The exponential growth of human population *will* stop. The only questions are *when* and *how*.

First consider the question of *when*. The highest estimates put the Earth's carrying capacity around 40 billion, which we would reach within about 150 years at recent growth rates. In that case, regardless of other assumptions, population growth must stop or slow considerably within the next 150 years—quite soon on the scale of human history.

As for *how*, there are only two basic ways to slow the growth of a population:

- a decrease in the birth rate or
- an increase in the death rate.

Most people are already choosing the first option, as birth rates now are at historic lows. Indeed, population actually is *decreasing* in a few European nations. Nevertheless, worldwide birth rates still are much higher than death rates, so exponential growth continues.

If a decrease in birth rates doesn't halt the exponential growth, an increase in the death rate will. If population significantly overshoots the carrying capacity by the time this process begins, the increase in the death rate will be dramatic—probably on a scale never before seen. This forecast is not a threat, a warning, or a prophecy of doom. It is simply a law of nature: *Exponential growth always stops.*

As human beings, we can choose to slow our population growth through intelligent and careful decisions. Or, we can choose to do nothing, leaving ourselves to the mercy of natural forces over which we have no more control than we do over hurricanes, tornadoes, earthquakes, or the explosions of distant stars. Either way, it's a choice that each and every one of us must make, and upon which our entire future depends.

The most fundamental question about population growth therefore concerns the carrying capacity. If the carrying capacity is well above the current population, then we have plenty of time to settle into logistic growth and long-term population stability. But if the current population is near the carrying capacity, we need to act quickly to prevent overshoot and collapse. Unfortunately, any estimate of carrying capacity is subject to great uncertainty, for at least four important reasons:

- The carrying capacity depends on consumption of resources such as energy. However, different countries consume at very different rates. For example, the carrying capacity is much lower if we assume that the growing population will consume energy at the U.S. average rate rather than the Japanese average rate (which is about half the U.S. rate).
- The carrying capacity depends on assumptions about the environmental impact of the average person. A larger average impact on the environment means a lower carrying capacity.
- The carrying capacity can change with both human technology and the environment. For example, estimates of carrying capacity typically consider the availability of fresh water. However, if we can develop new sources of energy (such as fusion), nearly unlimited amounts of fresh water may be obtained through the desalinization of seawater. Conversely, global warming might alter the environment and reduce our ability to grow food, thereby lowering the carrying capacity.
- Even if we could account for the many individual factors in the carrying capacity (such as food production, energy, and pollution), the Earth is such a complex system that precisely predicting the carrying capacity may well be impossible. For example, no one can predict whether or how much the loss of rain forest species affects the carrying capacity.

> The power of population is indefinitely greater than the power in the Earth to produce subsistence for man.
>
> —Thomas Malthus

The history of attempts to guess the carrying capacity of the Earth is full of missed predictions. Among the most famous was that made by English economist Thomas Malthus (1766–1834). In a 1798 paper entitled *An Essay on the Principle of Population as It Affects the Future Improvement of Society,* Malthus argued that food production would not be able to keep up with the rapidly growing populations of Europe and America. He concluded that mass starvation would soon hit these continents. His prediction did not come true, primarily because advances in technology *did* allow food production to keep pace with population growth.

Time Out to Think

Some people argue that while Malthus's immediate predictions didn't come true, his overall point about a limit to population is still valid. Others cite Malthus as a classic example of underestimating the ingenuity of our species. What do you think? Defend your opinion.

Case Study The Population of Egypt

Over long periods of time, real population growth patterns tend to be quite complex. Sometimes the growth may look exponential, while at other times it may appear logistic or like overshoot and collapse. It may even appear to be some combination of these possibilities all at once.

One of the few cases for which long-term population data are available is that of Egypt. Figure 8.6 shows these data, along with a few historical events that affected

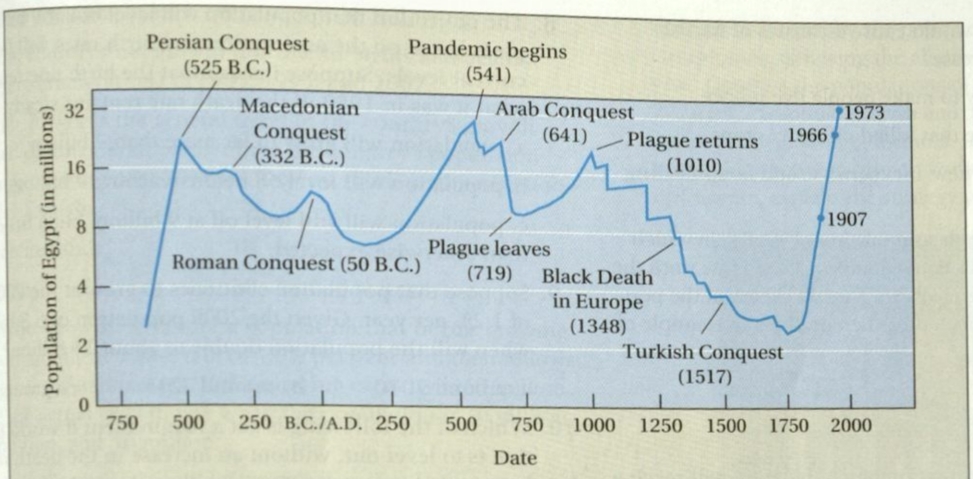


FIGURE 8.6 The historical population of Egypt. Notice that each tick mark on the vertical axis represents a doubling of the population in millions (2, 4, 8, 16, 32).

Source: T. H. Hollingsworth, *Historical Demography* (Ithaca, NY: Cornell University Press, 1969).

BY THE WAY

Current UN projections suggest that world population will level out later in this century at about 9 billion. These projections assume that the fertility rate will decline from the current level of 2.6 children per woman to the "replacement level" of 2.1 children per woman by 2050—an assumption that in turn is based on an assumption of greater availability of and education about contraceptive choices.

the population. (The graph uses an exponential vertical scale, with each tick mark representing a number twice as big as the previous one.) Note the complexity of the pattern. Even with the best mathematical models, it is hard to imagine that scientists in ancient Egypt could have predicted the future population of the region over a period of even a hundred years, let alone several thousand years.

This example offers an important lesson about mathematical models. They are useful for gaining insight into the processes being modeled. However, mathematical models can be used to *predict* future changes only when the processes are relatively simple. For example, it is easy to use mathematical modeling to predict the path of a spaceship because the law of gravity is relatively simple. But the growth of human population is such a complex phenomenon that we have little hope of ever being able to predict it reliably.

QUICK QUIZ

Choose the best answer to each of the following questions. Explain your reasoning with one or more complete sentences.

1. World population is currently rising by about 75 million people per year. About how many people are added to the population each minute?

   a. 5 b. 30 c. 140

2. Based on the data given in the text, world population in 2012 will be about

   a. 5.2 billion.
   b. 7.0 billion.
   c. 8.7 billion.

3. The primary reason for the rapid growth of human population over the past century has been

   a. an increasing birth rate. b. a decreasing death rate.
   c. a combination of an increasing birth rate and a decreasing death rate.

4. The *carrying capacity* of the Earth is defined as

   a. the maximum number of people who could fit elbow-to-elbow on the planet.
   b. the maximum population that could be sustained for a long period of time.
   c. the peak population that would be reached just before a collapse in the population size.

5. Which of the following would cause estimates of Earth's *carrying capacity* to increase?

   a. the discovery of a way to make people live longer

   b. the spread of a disease that killed off many crops

   c. the development of a new, inexpensive, and nonpolluting energy source

6. Recall the bacteria in a bottle example from Unit 8A, in which the number of bacteria in a bottle doubles each minute until the bottle is full and the bacteria all die. The full history of the population of these bacteria, including their death, is an example of

   a. overshoot and collapse.

   b. unending exponential growth.

   c. logistic growth.

7. When researchers project that human population will reach a steady 9 billion later in this century, what type of growth model are they assuming?

   a. overshoot and collapse

   b. exponential

   c. logistic

8. The projection that population will level out at 9 billion people is based on the assumption that birth rates will fall from current levels. Suppose instead that the birth rate returns to what it was in 1950. If the death rate remains steady,

   a. population will grow to far more than 9 billion.

   b. population will level off before reaching 9 billion.

   c. population will still level off at 9 billion, but a little sooner than otherwise expected.

9. Suppose that population continues to grow at the 2009 rate of 1.2% per year. Given the 2009 population of 6.8 billion, when will the population double to about 14 billion?

   a. around 2070 b. around 2215 c. around 2450

10. Which of the following is *not* a requirement if world population is to level out, without an increase in the death rate, later in this century at 9 or 10 billion people?

    a. We must increase food production by close to 50%.

    b. The average woman must give birth to fewer children than she does at present.

    c. We must discover a new source of inexpensive and nonpolluting energy.

Exercises 8C

REVIEW QUESTIONS

1. Based on Figure 8.3, contrast the changes in human population for the 10,000 years preceding C.E. 1 and the 2000 years since. What has happened over the past few centuries?

2. Briefly describe how the overall growth rate is related to birth and death rates.

3. How do today's birth and death rates compare to those in the past? Why is human population growing?

4. What do we mean by carrying capacity? Why is it so difficult to determine the carrying capacity of Earth?

5. What is logistic growth? Why would it be good if human population growth followed a logistic growth pattern in the future?

6. What is overshoot and collapse? Under what conditions does it occur? Why would it be a bad thing for the human race?

DOES IT MAKE SENSE?

Decide whether each of the following statements makes sense (or is clearly true) or does not make sense (or is clearly false). Explain your reasoning.

7. Within the next decade, world population will grow by more than twice the current population of the United States.

8. If birth rates fall more than death rates, the growth rate of world population will fall.

9. The carrying capacity of our planet depends only on our planet's size.

10. Thanks to rapid increases in computing technology, we should be able to pin down the carrying capacity of the Earth to a precise number within just a few years.

11. In the wild, we always expect the population of any animal species to follow a logistic growth pattern.

12. Past history gives us strong reason to believe that human population is following a logistic growth pattern.

BASIC SKILLS & CONCEPTS

13–16: Varying Growth Rates. Starting from a 2009 population of 6.8 billion, use the given growth rate to find the approximate doubling time (use the rule of 70) and to predict world population in 2050.

13. Use the average annual growth rate between 1850 and 1950, which was about 0.9%.

14. Use the average annual growth rate between 1950 and 2000, which was about 1.8%.

15. Use the average annual growth rate between 1970 and 2000, which was about 1.6%.

16. Use the current annual growth rate of the United States, which is about 0.7%.

17–20: Birth and Death Rates. The following table gives the birth and death rates for four countries in three different years.

	Birth Rate (per 1000)			Death Rate (per 1000)		
Country	1985	1995	2007	1985	1995	2007
Czech Rep.	14.5	9.3	9.0	11.8	11.4	10.6
Israel	23.8	21.0	17.7	6.6	6.3	6.2
Sweden	11.8	11.7	10.2	11.3	11.0	10.3
U.S.	15.7	15.1	14.2	8.7	8.8	8.3

For the country given in each exercise, do the following:

a. Find the country's net growth rate due to births and deaths (i.e., neglect immigration) in 1985, 1995, and 2007.

b. Describe in words the general trend in the country's growth rate. Based on this trend, predict how the country's population will change over the next 20 years. Do you think your prediction is reliable? Explain.

17. Czech Republic
18. Israel
19. Sweden
20. United States

21. **Logistic Growth.** Consider a population that begins growing exponentially at a base rate of 4.0% per year and then follows a logistic growth pattern. If the carrying capacity is 60 million, find the actual growth rate when the population is 10 million, 30 million, and 50 million.

22. **Logistic Growth.** Consider a population that begins growing exponentially at a base rate of 6.0% per year and then follows a logistic growth pattern. If the carrying capacity is 80 million, find the actual growth rate when the population is 10 million, 50 million, and 70 million.

FURTHER APPLICATIONS

23–26: **U.S. Population.** Starting from an estimated U.S. population of 305 million in 2009, use the given growth rate to estimate U.S. population in 2059 and 2109. Use the approximate doubling time formula.

23. Use the current U.S. annual growth rate of 0.7%.

24. Use a growth rate of 0.5%.

25. Use a growth rate of 1.0%.

26. Use a growth rate of 0.4%.

27. **Population Growth in Your Lifetime.** Starting from the 6.8 billion population in 2009, assume that world population maintains its current annual growth rate of 1.2%. What will be the world population when you are 50 years old? 80 years old? 100 years old?

28. **Slower Growth.** Repeat Exercise 27, but for a growth rate of 1% rather than 1.2%.

29–32: **World Carrying Capacity.** For the given carrying capacities, use the 1960s peak annual growth rate of 2.1% and population of 3 billion to predict the base growth rate and current growth rate with a logistic model. Assume a current world population of 6.8 billion. How do the predicted growth rates compare to the actual growth rate of about 1.2% per year?

29. Assume the carrying capacity is 8 billion.

30. Assume the carrying capacity is 10 billion.

31. Assume the carrying capacity is 15 billion.

32. Assume the carrying capacity is 20 billion.

33. **Growth Control Mediation.** A city with a 2010 population of 100,000 has a growth control policy that limits the increase in residents to 2% per year. Naturally, this policy causes a great deal of dispute. On one side, some people argue that *growth* costs the city its small-town charm and clean environment. On the other side, some people argue that *growth control* costs the city jobs and drives up housing prices. Finding their work limited by the policy, developers suggest a compromise of raising the allowed growth rate to 5% per year. Contrast the populations of this city in 2020, 2030, and 2070 for 2% annual growth and 5% annual growth. Use the approximate doubling formula. If you were asked to mediate the dispute between growth control advocates and opponents, explain the strategy you would use.

WEB PROJECTS

34. **Population Predictions.** Find population predictions from an organization that studies population, such as the United Nations or the U.S. Census Bureau. Read about how the predictions are made. Write a short summary of the methods used to predict future population. Be sure to discuss the uncertainties in the predictions.

35. **Carrying Capacity.** Find several different opinions concerning the Earth's human population carrying capacity. Based on your research, draw some conclusions about whether overpopulation presents an immediate threat. Write a short essay detailing the results of your research and clearly explaining your conclusions.

36. **U.S. Population Growth.** Research population growth in the United States to determine the relative proportions of the growth resulting from birth rates and from immigration. Then research both the problems and the benefits of the growing U.S. population. Form your own opinions about whether the United States has a population problem. Write an essay covering the results of your research and stating and defending your opinions.

37. **Thomas Malthus.** Find more information about Thomas Malthus and his famous predictions about population. Write a short paper on either his personal biography or his work.

38. **Extinction.** Choose an endangered species and research why it is in decline. Is the decline a case of overshoot and collapse? Is human activity changing the carrying capacity for the species? Write a short summary of your findings.

IN YOUR WORLD

39. **Population Growth.** Find a recent news story that concerns population growth. Does the story consider the long-term effects of the growth? If so, do you agree with the claims? If not, discuss a few possible effects of future growth.

40. **Immigration.** Within the United States, immigration is an important part of the overall population growth. Find a recent story that discusses either the pros or the cons of immigration. Discuss the story and its conclusions.