**Math 6320, Assignment 5** **Due: End of April**

1. Let $A$ be a finitely generated abelian group, viewed as an $\mathbb{Z}$-module. Describe $A_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ in $\mathbb{Z}$. Use the structure theorem for abelian groups.
2. Let $R$ be a ring. Verify that for any $r$ in $R$, the ring $R[x]/(xr-1)$ is (canonically isomorphic) to the localization of $R$ at the multiplicatively closed subset $\{r^i \mid i \geq 0\}$.
3. Let $I$ be an ideal in $R$ and $M$ an $R$-module such that $M_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m} \supseteq I$. Prove that $IM = M$.
4. Let $R$ be a ring and $M$ a faithful $R$-module; this means that $\operatorname{ann}_R M = (0)$. Prove that when $M$ is noetherian, as an $R$-module, the ring $R$ is noetherian.
5. Let $R$ be a Noetherian ring and $\varphi\colon R \to R$ a surjective homomorphism of rings. Is $\varphi$ an isomorphism?
6. Let $K$ be a field.
   (a) Suppose $f(x)$ in $K[x]$ has positive degree. Prove that $K[x]$ is a finitely generated $K[f(x)]$-module.
   (b) Let $R$ be a subring of $K[x]$ that contains $K$. Prove that $R$ is Noetherian.
   (c) Describe a non-noetherian subring of $K[x,y]$.
7. Suppose $K$ is not algebraically closed. Prove that each algebraic set in $K^n$ is the zero set of a single polynomial.
8. Prove that the subset $V = \{(t,t^2,...,t^n) \mid t \in \mathbb{C}\}$ of $\mathbb{C}^n$ is algebraic.
9. Let $K$ be an algebraically closed field and $L$ an extension field. If polynomials $f_1,\dots,f_c$ in $K[x_1,...,x_n]$ have a common root in $L^n$, prove they have a common root in $K^n$.
10. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{R}[x,y]$ containing $x^2+y^2+1$. What is the quotient $\mathbb{R}[x,y]/\mathfrak{m}$?