Math 6320, Assignment 5  
Due: End of April

1. Let $A$ be a finitely generated abelian group, viewed as an $\mathbb{Z}$-module. Describe $A_p$ for each prime ideal $p$ in $\mathbb{Z}$. Use the structure theorem for abelian groups.

2. Let $R$ be a ring. Verify that for any $r$ in $R$, the ring $R[x]/(xr - 1)$ is (canonically isomorphic) to the localization of $R$ at the multiplicatively closed subset $\{r^i \mid i \geq 0\}$.

3. Let $I$ be an ideal in $R$ and $M$ an $R$-module such that $M_m = 0$ for each maximal ideal $m \supseteq I$. Prove that $IM = M$.

4. Let $R$ be a ring and $M$ a faithful $R$-module; this means that $\operatorname{ann}_R M = (0)$. Prove that when $M$ is noetherian, as an $R$-module, the ring $R$ is noetherian.

5. Let $R$ be a Noetherian ring and $\varphi : R \to R$ a surjective homomorphism of rings. Is $\varphi$ an isomorphism?

6. Let $K$ be a field.
   (a) Suppose $f(x)$ in $K[x]$ has positive degree. Prove that $K[x]$ is a finitely generated $K[f(x)]$-module.
   (b) Let $R$ be a subring of $K[x]$ that contains $K$. Prove that $R$ is Noetherian.
   (c) Describe a non-Noetherian subring of $K[x,y]$.

7. Suppose $K$ is not algebraically closed. Prove that each algebraic set in $K^n$ is the zero set of a single polynomial.

8. Prove that the subset $V = \{(t,t^2,...,t^n) \mid t \in \mathbb{C}\}$ of $\mathbb{C}^n$ is algebraic.

9. Let $K$ be an algebraically closed field and $L$ an extension field. If polynomials $f_1, \ldots, f_c$ in $K[x_1, \ldots, x_n]$ have a common root in $L^n$, prove they have a common root in $K^n$.

10. Let $m$ be a maximal ideal of $\mathbb{R}[x,y]$ containing $x^2 + y^2 + 1$. What is the quotient $\mathbb{R}[x,y]/m$?