Math 6320, Assignment 3 **Due: Weekend of March 4th**

1. For each of the following polynomials, determine the number of distinct roots in $\mathbb{F}_{49}$.

$$x^{48}-1, \qquad x^{49}-1, \qquad x^{54}-1.$$

2. Let k be a field of characteristic $p > 0$. Set $E = k(x,y)$ and $F = k(x^p, y^p)$.
   (a) Prove that $[E : F] = p^2$.
   (b) Prove that $E \neq F(\alpha)$ for any $\alpha \in E$.
3. Prove that the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is normal (also called Galois), and compute its Galois group.
4. Determine the Galois group of $x^4 - 2 \in \mathbb{Q}[x]$.
5. Determine the Galois group of $x^4 + 4x^2 + 2 \in \mathbb{Q}[x]$.
6. Compute the cyclotomic polynomials $\Phi_{30}(x)$ and $\Phi_{81}(x)$. Use the formula given by Möbius inversion.
7. This exercise and the next give a quick method for computing cyclotomic polynomials.
   (a) Prove that $\Phi_n(x) = x^{\varphi(n)}\Phi_n(1/x)$, so that the coefficients of $\Phi_n(x)$ are palindromic.
   (b) Prove that $\Phi_n(x)$ is determined by its value in the residue ring $\mathbb{Z}[x]$ modulo $(x^{\lceil \varphi(n)/2 \rceil + 1})$.
8. Prove the following identities.
   (a) $\Phi_n(x) = \Phi_m(x^{n/m})$ when m is the product of the distinct prime factors dividing n.
   (b) $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$ when p is a prime not dividing n.
   (c) $\Phi_{2n}(x) = \Phi_n(-x)$ when n is an odd integer ≥ 3.
9. Use the preceding exercises to compute $\Phi_n(x)$ for $n = 30, 81, 105$; compare with your answer for (6).