Math 6320, Assignment 1 **Due: Weekend of January 26**

1. Express the symmetric polynomial $x^2y^2 + y^2z^2 + z^2x^2$ in terms of elementary symmetric polynomials in x, y, z.

2. Set $\mathbb{Z}[\underline{x}] := \mathbb{Z}[x_1, \dots, x_n]$ and for each integer $d \geq 1$ consider the symmetric polynomial

$$p_d(\underline{x}) := \sum_{i=1}^{n} x_i^d \,.$$

Let $\mathbb{Z}[|\underline{x}|] := \mathbb{Z}[|x_1, \dots, x_n|]$ be the ring of formal power series in $\underline{x}$, and consider the map $\delta \colon \mathbb{Z}[|\underline{x}|] \to \mathbb{Z}[|\underline{x}|]$ defined as follows:

$$\delta(\sum_{i=0}^{\infty} u_i) := \sum_{i=0}^{\infty} i u_i \,.$$

This exercise leads to a recursive relation (due to Newton) for expressing $p_d(\underline{x})$ in terms of $s_k(\underline{x})$, the elementary symmetric polynomials.

   (a) Prove that δ is a $\mathbb{Z}$-linear map satisfying the Leibniz rule $\delta(uv) = \delta(u)v + u\delta(v)$; said otherwise, δ is a $\mathbb{Z}$-derivation of $\mathbb{Z}[|\underline{x}|]$.

   (b) Verify the following identity

$$\frac{\delta\left(\prod_{i=1}^{n}(1+x_i)\right)}{\prod_{i=1}^{n}(1+x_i)} = \frac{x_1}{1+x_1} + \cdots + \frac{x_n}{1+x_n}$$

   (c) Prove that $\prod_{i=1}^{n}(1+x_i) = \sum_{i=1}^{n} s_i$.

   (d) Compute $\delta\left(\prod_{i=1}^{n}(1+x_i)\right)$ using (c) and compare it what one gets (b) to deduce that, for $d \geq 1$, one has

$$p_d(\underline{x}) = (-1)^{d-1} d s_d(\underline{x}) + \sum_{k=1}^{d-1} (-1)^{k-1} s_k(\underline{x}) p_{d-k}(\underline{x}) \,.$$

   Recall that $s_k(\underline{x}) = 0$ for $k > n$.

3. Using the preceding exercise, compute the sum of the seventh powers of the roots of the equation $y^3 + py + q$, where p, q are integers.

4. In the following exercise $D(f)$ denotes the discriminant of a polynomial $f(X)$.

   (a) If $f(X) = (X-a)g(X)$, prove that $D(f) = g(a)^2 D(g)$.

   (b) Compute $D(X^n - 1)$, for $n \geq 1$, and $D(X^{n-1} + X^{n-2} + \cdots + 1)$, for $n \geq 2$.

5. This exercise derives an expression for the resultant in terms of determinants. Consider the following polynomials in $\mathbb{Z}[\underline{x}, \underline{y}][X]$, the polynomials in the indeterminate X with coefficients in the ring $\mathbb{Z}[\underline{x}, \underline{y}]$.

$$f(X) := \prod_{i=1}^{m}(X - x_i) = \sum_{i=0}^{m} a_i X^i \quad \text{and} \quad g(X) := \prod_{j=1}^{n}(X - y_j) = \sum_{j=0}^{n} b_j X^i$$

Consider the $(m+n) \times (m+n)$ matrices; the one on the left is a Vandermonde matrix.

$$V := \begin{bmatrix} y_1^{m+n-1} & y_1^{m+n-2} & \cdots & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ y_n^{m+n-1} & y_n^{m+n-2} & \cdots & y_n & 1 \\ x_1^{m+n-1} & x_1^{m+n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ x_n^{m+n-1} & x_m^{m+n-2} & \cdots & x_m & 1 \end{bmatrix} \quad \text{and} \quad A := \begin{bmatrix} a_m & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{m-1} & a_m & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & 0 \\ a_0 & 0 & \cdots & a_m & b_0 & 0 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{m-1} & 0 & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & a_0 & \vdots & \vdots & \vdots & b_0 \end{bmatrix}$$

   (a) Compute $\det(VA)$ in two different ways to deduce that $\operatorname{Res}_X(f,g) = \det(A)$. For this, you will need (and can use without proof) the standard computation of the determinant of a Vandermonde matrix.

   (b) Deduce that $\operatorname{Res}_X(f,g)$ is in $\mathbb{Z}[a_0, \dots, a_m, b_0, \dots, b_n]$; in words, this means that the resultant can be expressed as a polynomial in the coefficients of $f(X)$ and $g(X)$.