Math 6310, Assignment 6 **Due in class: Monday, November 30**

1. Determine, up to isomorphism, all finitely generated subgroups of $\mathbb{Q}$.

2. Determine all maximal ideals of the ring $\mathbb{Z}[x]/(120, x^3+1)$.

3. Let M be a $\mathbb{Z}$-module that is a homomorphic image of $\mathbb{Z}^2$, such that each element of M is annihilated by 6. Determine all such M up to isomorphism.

4. Let $\mathbb{F}$ be a field, and $R = \mathbb{F}[x]$ a polynomial ring. Let M be the quotient of R^3 by the submodule generated by the elements (x^2, x^3, x^4) and $(1, x+1, x^2)$ of R^3. Express M as a direct sum of cyclic R-modules.

5. If M is a square matrix over $\mathbb{C}$ with $M^3 = M$, prove that M is diagonalizable. Is this true over arbitrary fields?

6. Find the Jordan form of the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

7. Find the possible Jordan forms of a real 3×3 matrix M satisfying $M^2 + M + I = 0$.

8. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Determine the possible Jordan forms of the matrix
$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

9. Determine the number of conjugacy classes of elements of order 2 in $\mathrm{GL}_n(\mathbb{F})$. Does this depend on the characteristic of the field $\mathbb{F}$?

10. If M is a 5×5 complex matrix with $\det(M) = 1$, $\mathrm{trace}(M) = 5$, $\mathrm{rank}(M - I) = 3$, and minimal polynomial having degree 3, find the possible Jordan forms of M.