1. If \(1 - ab\) is invertible in a ring, show that \(1 - ba\) is also invertible.

2. Let \(R\) be a ring in which \(x^3 = x\) for each \(x\). Prove that \(R\) is commutative.

   **In the problems below, \(R\) is a commutative ring.**

3. In which of the following rings is every ideal principal? Justify your answer.
   
   (a) \(\mathbb{Z} \times \mathbb{Z}\),  
   (b) \(\mathbb{Z}/4\),  
   (c) \((\mathbb{Z}/6)[x]\),  
   (d) \((\mathbb{Z}/4)[x]\).

4. If \(R\) is a domain that is not a field, prove that the polynomial ring \(R[x]\) is not a principal ideal domain.

5. An element \(r\) in a ring \(R\) is **nilpotent** if \(r^n = 0\) for some \(n \geq 0\). Prove the following assertions.
   
   (a) If \(r\) is nilpotent, then \(1 + r\) is invertible in \(R\).
   
   (b) If \(r_1, \ldots, r_c\) are nilpotent elements, then any element in the ideal \((r_1, \ldots, r_c)\) is nilpotent.

6. Let \(R\) be a commutative ring and \(R[x]\) the polynomial ring over \(R\) in the indeterminate \(x\). Let
   
   \[ f(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n \quad \text{with} \quad r_i \in R. \]

   Prove the following assertions.
   
   (a) \(f(x)\) is nilpotent if and only if \(r_0, \ldots, r_n\) are nilpotent.
   
   (b) \(f(x)\) is a unit in \(R[x]\) if and only if \(r_0\) is a unit in \(R\) and \(r_1, \ldots, r_n\) are nilpotent.
   
   (c) \(f(x)\) is a zerodivisor if and only if there exists a nonzero element \(r \in R\) such that \(r \cdot f = 0\).

   Recall that \(a \in R\) is a **zerodivisor** if there exists \(b \neq 0\) in \(R\) with \(ab = 0\); the only zerodivisor in a domain is 0.