Math 6310, Assignment 1
Due in class: Friday, September 4

1. Suppose $G$ is a finite set with an associative law of composition, and $e \in G$ is an element such that $xe = x = ex$ for all $x \in G$. If $G$ has the property that

$$xz = yz \implies x = y,$$

prove that $G$ is a group.

2. Let $G$ be a group, and let $H$ be a subgroup of finite index. Prove that the number of right cosets of $H$ equals the number of left cosets.

3. Let $a$ be an element of a group $G$. Prove that there exists $x$ in $G$ with $x^2 ax = a^{-1}$ if and only if $a$ is the cube of some element of $G$.

4. Let $G$ be a finite group. If $x^2 = e$ for each $x \in G$, prove that $|G|$ is a power of 2.

5. Find a group with elements $a, b$ such that $a$ and $b$ have finite order, but $ab$ does not have finite order. (Hint: Try looking in $GL_2(\mathbb{Z})$, the group of invertible $2 \times 2$ matrices over $\mathbb{Z}$.)

6. Let $n$ be a positive integer. Consider the set $G$ of positive integers less than or equal to $n$ that are relatively prime to $n$. The number of elements of $G$ is the Euler phi-function, denoted $\phi(n)$.

   (a) Show that $G$ is a group under multiplication modulo $n$.

   (b) If $m$ and $n$ are relatively prime positive integers, show that $m^{\phi(n)} \equiv 1 \pmod{n}$.

7. Let $n$ be a positive integer. Show that $n = \sum_{d | n} \phi(d)$, where the sum is taken over all positive integers $d$ that divide $n$. (Hint: A cyclic group of order $n$ has a unique subgroup of order $d$ for each $d$ dividing $n$.)

8. Let $G$ be a finite group with the property that for each integer $d \geq 1$, the equation $x^d = e$ has at most $d$ solutions in $G$. Prove that $G$ is cyclic.

9. Let $G$ be a group such that for a fixed integer $n > 1$, we have $(xy)^n = x^n y^n$ for all $x, y \in G$. Let

$$G^{(n)} = \{ x^n \mid x \in G \} \quad \text{and} \quad G_{(n)} = \{ x \in G \mid x^n = e \}.$$

   (a) Prove that $G^{(n)}$ and $G_{(n)}$ are normal subgroups of $G$.

   (b) If $G$ is finite, show that the order of $G^{(n)}$ equals the index of $G_{(n)}$.

   (c) Show that for all $x, y \in G$, we have $x^{1-n} y^{1-n} = (xy)^{1-n}$. Use this to get $x^{n-1} y^n = y^n x^{n-1}$.

   (d) Conclude that elements of $G$ of the form $x^{n(n-1)}$ generate an abelian subgroup.

10. Let $G$ be a group such that $(xy)^3 = x^3 y^3$ for all $x, y \in G$, and such that the map $x \mapsto x^3$ is bijective. Prove that the group $G$ is abelian.