MATH 2270-2, SPRING 2006, PRACTICE EXAM 3

1. Let M_n be the $n \times n$ matrix with all 2's along "the other diagonal", and 0's everywhere else. For example,

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

Let $d_n = \det(M_n)$.

(1) Find a formula expressing d_n in terms of d_{n-1} for positive integers $n \geq 2$.

(2) Find $d_1, d_2, \ldots, d_8$. Do you see a pattern? Find a closed formula for d_n and justify your formula by the mathematical induction.

(3) Find d_{100}.

2. Let $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}$ be the 3×3 matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Suppose $\det(A) = 3$. Find the following.

(1) $\det\left(\begin{bmatrix} -\vec{v}_2 \\ -2\vec{v}_1 \\ \vec{v}_3 \end{bmatrix}\right) =$

(2) $\det\left(\begin{bmatrix} -\vec{v}_2 \\ -2\vec{v}_3 \\ \vec{v}_1 \end{bmatrix}\right) =$

(3) $\det\left(\begin{bmatrix} -2\vec{v}_2 + 3\vec{v}_3 \\ -2\vec{v}_1 \\ 6\vec{v}_2 - 9\vec{v}_3 \end{bmatrix}\right) =$

3. Consider the parallelepiped V in $\mathbb{R}^3$ defined by three vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

in $\mathbb{R}^3$.

(1) Find the volume of the parallelepiped V.

(2) What is the volume of the image $T(V)$ of V under the linear transformation $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix} \vec{x}$.

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4. Consider the linear transformation $T : P_2 \to P_2$ defined by two cases:

(1) $T(f(x)) = f(2x-1)$

(2) $T(f(x)) = f'(x)$

Let $\mathcal{B} = (1, x, x^2)$ be the standard basis of P_2. Answer the following questions for each of maps above.

(a) Find the $\mathcal{B}$-matrix A of T

(b) the Find the determinant of T.

(c) Find all eigenvalues and all eigenvectors of T. Specify the algebraic multiplicity and geometric multiplicity of each eigenvalue.

(d) Determine whether P_2 has an eigenbasis of T. If P_2 has an eigenbasis of T, let $\mathcal{D}$ be an eigenbasis and find the $\mathcal{D}$-matrix B of T.

⟨ True or False questions ⟩

Determine whether the following statement is True or False.

(1) If 0 is an eigenvalue of a matrix A, then $\det(A) = 0$.

(2) If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^3 as well.

(3) The matrix of any orthogonal projection on a line L in $\mathbb{R}^2$ gives an eiganbasis of $\mathbb{R}^2$.

(4) If an invertible matrix A gives an eigenbasis of $\mathbb{R}^n$, then A^{-1} must give an eigenbasis of $\mathbb{R}^n$ as well.

(5) If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of A, then $\vec{v} + \vec{w}$ is also an eigenvector of A.

(6) $\det(AB)=\det(A)\det(B)$.

(7) $\det(A+B)=\det(A)+\det(B)$.

(8) $\det(AB)=\det(BA)$.

(9) If all the entries of a 7×7 matrix A are 7, then $\det(A)$ must be 7^7.

(10) If the determinant of an 5×5 matrix A is 5, then its rank must be 5.

(11) If A is any symmetric matrix, then $\det(A) = 1$ or -1.

(12) If A is any skew-symmetric 4×4 matrix, then $\det(A) = 0$.

(13) If A is any skew-symmetric 5×5 matrix, then $\det(A) = 0$.

(14) If A is orthogonal, then $\det(A) = 1$ or -1.

(15) There exists invertible 3×3 matrix A and S such that $S^{-1}AS = -A$.

(16) There exists a 3×3 matrix A such that $A^2 = -I_3$.

(17) If an $n \times n$ matrix A is invertible, then $adj(A)$ is invertible as well.

(18) If A is a 6×6 matrix with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and E_{λ_1} has geometric multiplicity 2 and E_{λ_2} has geometric multiplicity 3. Then $\mathbb{R}^6$ has an eigenbasis of A.

Good-luck!