<A part of lecture notes on Feb. 21, 2006>

1. Change of bases matrix from $\mathcal{B}$ to $\mathcal{U}$, where $\mathcal{B}$ & $\mathcal{U}$ are bases of V.

Let $\mathcal{B}=(f_1,\dots,f_n)$ be a basis of V.

Then $\boxed{\begin{matrix} S_{\mathcal{B}\to\mathcal{U}} \\ \text{(Change of bases.)}\end{matrix}} = \begin{bmatrix} [f_1]_{\mathcal{U}} & [f_2]_{\mathcal{U}} & \cdots & [f_n]_{\mathcal{U}} \end{bmatrix}$. & $S_{\mathcal{B}\to\mathcal{U}}([f]_{\mathcal{B}}) = [f]_{\mathcal{U}}$.

2. the $\mathcal{B}$-matrix of T:

Let $\mathcal{B}=(f_1,\dots,f_n)$ be a basis of V & $T: V\to V$, a linear map.

Then $\boxed{\text{the } \mathcal{B}\text{-matrix}^{B} \text{ of } T} = \begin{bmatrix} [T(f_1)]_{\mathcal{B}} & [T(f_2)]_{\mathcal{B}} & \cdots & [T(f_n)]_{\mathcal{B}} \end{bmatrix}$.

3. Change of bases matrix & $\mathcal{B}$-matrix / $\mathcal{U}$-matrix.

Let A be the $\mathcal{U}$-matrix of T, B the $\mathcal{B}$-matrix of T

and $S=S_{\mathcal{B}\to\mathcal{U}}$ the change of bases from $\underline{\mathcal{B} \text{ to } \mathcal{U}}$. $\leftarrow$ corrected!

Then $\boxed{AS = SB}$ $\Rightarrow B = S^{-1}AS$

Ex. Let $T: P_2 \to P_2$ by $T(f) = f' + f(0)$. & let $\mathcal{U}=(1,t,t^2)$ & $\mathcal{B}=(1,t-1,t^2-t)$ be bases of P_2.

Then $S = S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} [1]_{\mathcal{U}} & [t-1]_{\mathcal{U}} & [t^2-t]_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $\leftarrow$ corrected!

the $\mathcal{U}$-matrix $A = \begin{bmatrix} [T(1)]_{\mathcal{U}} & [T(t)]_{\mathcal{U}} & [T(t^2)]_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} [1]_{\mathcal{U}} & [1]_{\mathcal{U}} & [2t]_{\mathcal{U}} \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

So the $\mathcal{B}$-matrix $B = S^{-1}AS = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$.

In fact, $B = \begin{bmatrix} [T(1)]_{\mathcal{B}} & [T(t-1)]_{\mathcal{B}} & [T(t^2-t)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} [1]_{\mathcal{B}} & [0]_{\mathcal{B}} & [2t-1]_{\mathcal{B}} \end{bmatrix}$

$= \underline{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$ So we get the same answer!