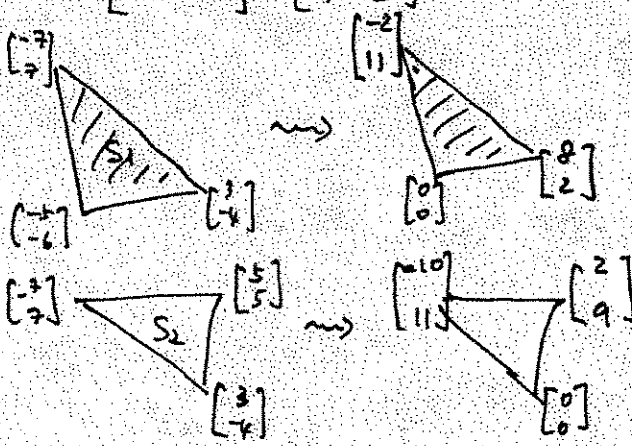
Sec 6.3

HW 9. Solution

#2. $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} \Rightarrow S = \frac{1}{2}\left|\det A\right| = \frac{1}{2}\left|(6-56)\right| = 25$

#7.



$\Rightarrow S_1 = \frac{1}{2}\left|\det\begin{bmatrix} -2 & 8 \\ 13 & 2 \end{bmatrix}\right| = \frac{1}{2}(108) = 54$.

$\Rightarrow S_2 = \frac{1}{2}\left|\det\begin{bmatrix} -10 & 2 \\ 11 & 9 \end{bmatrix}\right| = \frac{1}{2}(112) = 56$.

So  $= S_1 + S_2 = 54 + 56 = 110$.

#24. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$.

$\det(A) = \det\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} = \det\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 0 & -9 & 7 \end{bmatrix} = 2\cdot\det\begin{bmatrix} 4 & 5 \\ -9 & 7 \end{bmatrix} = 2\cdot(28+45) = 146$

$$x = \frac{\det\begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{\det(A)} = \frac{-\det\begin{bmatrix} -1 & 0 & 7 \\ 3 & 4 & 5 \\ 8 & 3 & 0 \end{bmatrix}}{146} = \frac{+1}{146}\det\begin{bmatrix} 1 & 0 & -7 \\ 3 & 4 & 5 \\ 8 & 3 & 0 \end{bmatrix} = \frac{1}{146}\det\begin{bmatrix} 1 & 0 & -7 \\ 0 & 4 & 26 \\ 0 & 3 & 56 \end{bmatrix}$$

$$= \frac{\det\begin{bmatrix} 4 & 26 \\ 3 & 56 \end{bmatrix}}{146} = \frac{146}{146} = 1$$

$$y = \frac{\det\begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{\det(A)} = \frac{\det\begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & -25 & 7 \end{bmatrix}}{146} = \frac{2\cdot\det\begin{bmatrix} 3 & 5 \\ -25 & 7 \end{bmatrix}}{146} = \frac{2\cdot(21+125)}{146} = 2$$

$$z = \frac{\det\begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{\det(A)} = \frac{\det\begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 0 & -9 & -25 \end{bmatrix}}{146} = \frac{2\cdot\det\begin{bmatrix} 4 & 3 \\ -9 & -25 \end{bmatrix}}{146} = \frac{2\cdot(-100+27)}{146} = -1$$.

So $\boxed{x = 1,\quad y = 2,\quad z = -1}$

#25. $$\mathrm{adj}(A) = \begin{bmatrix} +\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -\det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ -\det\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} & -\det\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} & -\det\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\cdot\mathrm{adj}(A) = \frac{1}{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

#26. adj(A) has entries coming from $\pm\det(A_{ji})$ so it has integer entries.

$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \frac{1}{1}\,\mathrm{adj}(A) = \mathrm{adj}(A)$ has also integer entries. //

#30. $$\mathrm{adj}(A) = \begin{bmatrix} \det\begin{bmatrix}3&0\\5&6\end{bmatrix} & -\det\begin{bmatrix}0&0\\5&6\end{bmatrix} & \det\begin{bmatrix}0&0\\3&0\end{bmatrix} \\ -\det\begin{bmatrix}2&0\\4&6\end{bmatrix} & \det\begin{bmatrix}1&0\\4&6\end{bmatrix} & -\det\begin{bmatrix}1&0\\2&0\end{bmatrix} \\ \det\begin{bmatrix}2&3\\4&5\end{bmatrix} & -\det\begin{bmatrix}1&0\\4&5\end{bmatrix} & \det\begin{bmatrix}1&0\\2&3\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix}$$ //

#34. We know $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) \Rightarrow I_n = \frac{1}{\det(A)} \cdot A(\mathrm{adj}(A))$

$\Rightarrow A\cdot(\mathrm{adj}(A)) = \boxed{\det(A)\cdot I_n}$

$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) \Rightarrow \det(A)\cdot A^{-1} = \mathrm{adj}(A) \Rightarrow \det(A)\cdot I_n = \mathrm{adj}(A)(A)$.

So the same as $A(\mathrm{adj}(A)) = (\mathrm{adj}(A))A = \boxed{\det(A)\cdot I_n}$, //

#35. $$\det(A^{-1}) = \det\left(\frac{1}{\det(A)}\cdot \mathrm{adj}(A)\right) = \frac{1}{(\det(A))^n}\det(\mathrm{adj}(A)).$$

Since $\det(A^{-1}) = \frac{1}{\det(A)}$, $\frac{1}{\det(A)} = \frac{1}{\det(A)^n}\cdot\det(\mathrm{adj}(A))$.

$$\Rightarrow \boxed{(\det(A))^{n-1} = \det(\mathrm{adj}(A)),}$$ //

Sec 7.1.

#2. We need to show that $\lambda \neq 0$, since A is invertible. Let $A\vec{v} = \lambda\vec{v}$.
If $\lambda = 0$, then $A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \vec{0} \Rightarrow I_n\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$ which is not true, since $\vec{v}$ is an eigenvector so nonzero.
So $\underline{\lambda \neq 0}$. Then from $A\vec{v} = \lambda\vec{v}$, $A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$.
$\Rightarrow I_n\vec{v} = \lambda A^{-1}\vec{v} \Rightarrow \vec{v} = \lambda A^{-1}\vec{v} \Rightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$. So $\vec{v}$ is an eigenvector of A^{-1} associated to $\frac{1}{\lambda}$.

#4. $7A\vec{v} = 7\lambda\vec{v} \Rightarrow (7A)\vec{v} = (7\lambda)\vec{v}$. $\Rightarrow \vec{v}$ is an eigenvector of $7A$ with 7λ.

#5. From $A\vec{v} = \lambda\vec{v}$ & $B\vec{v} = \mu\vec{v}$ for some scalars λ & μ,
$(A+B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \mu\vec{v} = (\lambda+\mu)\vec{v}$. So it is an eigenvector of $A+B$ associated to $(\lambda+\mu)$ which is the sum of two eigenvalues.

#6. From $A\vec{v} = \lambda\vec{v}$ & $B\vec{v} = \mu\vec{v}$, $(AB)\vec{v} = A(B\vec{v}) = A(\mu\vec{v}) = \mu(A\vec{v}) = (\mu\lambda)\vec{v}$
So $\vec{v}$ is an eigenvector of AB associated to $(\lambda\mu)$ which is the product of two eigenvalues.

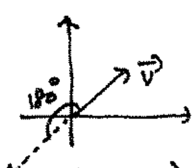
#10. let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ s.t. $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} a+2b=5 \\ c+2d=10 \end{cases} \Rightarrow \begin{cases} a=5-2b \\ c=10-2d \end{cases}$
If A looks like $\begin{bmatrix} 5-2b & b \\ 10-2d & d \end{bmatrix}$ for all $\underline{b \,\&\, d}$, then it has eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ w.r.t. $\lambda = 5$.

#13. Find $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $A\begin{bmatrix} a \\ b \end{bmatrix} = 4\begin{bmatrix} a \\ b \end{bmatrix}$. $\begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4a \\ 4b \end{bmatrix}$ * important *

$\Rightarrow \left.\begin{matrix} -6a+6b=4a \\ -15a+13b=4b \end{matrix}\right\} \Rightarrow \begin{cases} 10a=6b \\ -15a=-9b \end{cases} \Rightarrow a = \frac{3}{5}b. \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = b\begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$ $\boxed{\text{for } b \neq 0}$ //

So all eigenvectors of A associated with 4 are nonzero multiples of $\begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$ //
(or $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$).

#16.



$T(\vec{v}) = -\vec{v}$ for all nonzero vector $\vec{v}$.

So eigenvectors are all nonzero vectors associated to $\underline{\underline{\lambda = -1}}$.

#36. Find $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ s.t. $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ & $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 10\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
$\begin{cases} 3a+b=15 \\ 3c+d=5 \end{cases}$ & $\begin{cases} a+2b=10 \\ c+2d=20 \end{cases} \Rightarrow a=4, b=3, c=-2, d=11$.
So $\boxed{A = \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}}$ //

Sec 7.2

#4. $f_A(\lambda) = \det\begin{bmatrix}0-\lambda & 4\\ -1 & 4-\lambda\end{bmatrix} = -\lambda(4-\lambda)+4 = \lambda^2-4\lambda+4 = \boxed{(\lambda-2)^2.}$

$\Rightarrow$ $\underline{\lambda=2}$ with a.m(2) = 2.

#8. $f_A(\lambda) = \det\begin{bmatrix}-1-\lambda & -1 & -1\\ -1 & -1-\lambda & -1\\ -1 & -1 & -1-\lambda\end{bmatrix} = (-1-\lambda)\det\begin{bmatrix}-1-\lambda & -1\\ -1 & -1-\lambda\end{bmatrix} + 1\cdot\det\begin{bmatrix}-1 & -1\\ -1 & -1-\lambda\end{bmatrix} - 1\cdot\det\begin{bmatrix}-1 & -1\\ -1-\lambda & -1\end{bmatrix}$

$= (-1-\lambda)\left((-1-\lambda)^2-1\right) + \left(-(-1-\lambda)-1\right) - \left(1+(-1-\lambda)\right)$

$= (-1-\lambda)(\lambda^2+2\lambda) + (\lambda) + \lambda$

$= \lambda(\lambda+2)(-\lambda-1)+2\lambda = \lambda\left((\lambda+2)(-\lambda-1)+2\right) = \lambda(-\lambda^2-3\lambda)$

$= -\lambda(\lambda(\lambda+3))$

So $= -\lambda^2(\lambda+3)$

$\Rightarrow$ $\lambda=0$ with a.m.=2, & $\lambda=-3$ with a.m.=1

#10. $f_A(\lambda) = \det\begin{bmatrix}-3-\lambda & 0 & 4\\ 0 & -1-\lambda & 0\\ -2 & 7 & 3-\lambda\end{bmatrix} = (-3-\lambda)\det\begin{bmatrix}-1-\lambda & 0\\ 7 & 3-\lambda\end{bmatrix} + (-2)\cdot\det\begin{bmatrix}0 & 4\\ -1-\lambda & 0\end{bmatrix}$

$= -(\lambda+3)\left((-1-\lambda)(3-\lambda)\right) - 2\cdot 4(1+\lambda)$

$= -(\lambda+3)(\lambda-3)(\lambda+1) - 8(\lambda+1)$

$= -(\lambda+1)\left((\lambda+3)(\lambda-3)+8\right)$

$= -(\lambda+1)(\lambda^2-9+8)$

$= -(\lambda+1)(\lambda^2-1) = -(\lambda+1)^2(\lambda-1)$

$\Rightarrow$ $\lambda=1$ with a.m.=1 & $\lambda=-1$ with a.m = 2.

#14. $f_A(\lambda) = \det\begin{bmatrix}B-\lambda I_2 & C\\ 0 & D-\lambda I_2\end{bmatrix} \underset{\text{(Fact 6.1.8)}}{=} \det(B-\lambda I_2)\det(D-\lambda I_2) = f_B(\lambda)\cdot f_D(\lambda)$.

$\Rightarrow$ The eigenvalues of A are eigenvalues of B & D. (C doesn't help.).

& algebraic multiplicity of λ of A is the sum of a.m(λ) of B & a.m(λ) of D.

#15. $f_A(\lambda) = \det\begin{bmatrix}1-\lambda & k\\ 1 & 1-\lambda\end{bmatrix} = (1-\lambda)^2-k = \underline{\lambda^2-2\lambda+1-k} = 0$

This quadratic equation has two distinct real zeros iff $(-2)^2-4(1-k)>0$.

iff $\boxed{k>0.}$ iff two distinct eigenvalues exist.

This has no real eigenvalues iff $(-2)^2-4(1-k)<0$ iff $k<0.$

iff has no real eigenvalues.

Sec 7.2 (Continued)

\# 16. $f_A(\lambda) = \det \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2.$

it has two distinct eigenvalues iff $(a+c)^2 - 4(ac-b^2) > 0$

iff $a^2 - 2ac + c^2 + 4b^2 > 0$

$\quad\;\; \overset{\shortparallel}{(a-c)^2 + 4b^2} > 0.$ Since $b \neq 0$, $4b^2 > 0$ & $(a-c)^2 \geq 0$. So this inequality is always true.

So for any nonzero constants a, b & c, A has two distinct eigenvalues.

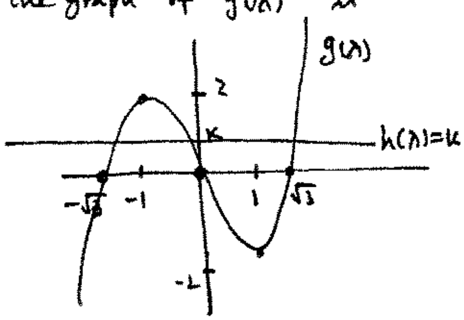
\#32. $f_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ k & 3 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} -\lambda & 1 \\ 3 & -\lambda \end{bmatrix} + k \det \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}$

$= -\lambda(\lambda^2 - 3) + k(1) = \underline{-\lambda(\lambda^2-3) + k}$

let $g(\lambda) = \lambda(\lambda^2-3)$ & $h(\lambda) = k$.

If $g(\lambda) = h(\lambda) \Rightarrow f_A(\lambda) = h(\lambda) - g(\lambda)$. So in order for $f_A(\lambda)$ to have 3 distinct real eigenvalues, $g(\lambda)$ & $h(\lambda)$ intersect at three real λ-values.

The graph of $g(\lambda)$ is



& it has local max 2 at $\lambda = -1$

& local min -2 at $\lambda = 1$.

So if $-2 < k < 2$, then $g(\lambda)$ intersects $h(\lambda) = k$ at three distinct values.

$\Rightarrow \boxed{-2 < k < 2}$ //

\#38. $f_A(\lambda) = (-\lambda)^2 + tr(A)(-\lambda) + \det(A)$

$= \lambda^2 - 5\lambda - 14 = (\lambda-7)(\lambda+2) \Rightarrow \lambda = 7, -2$ with a.m = 1 each.

\#42. $tr((A+B)^2) = tr(A^2 + AB + BA + B^2)$ ($\leftarrow$ here you shouldn't write $A^2 + 2AB + B^2$)

$= tr(A^2) + tr(AB) + tr(BA) + tr(B^2)$

$= tr(A^2) + tr(BA) + tr(BA) + tr(B^2)$ since $tr(AB) = tr(BA)$.

$= tr(A^2) + tr(B^2) + 2 \cdot tr(0)$ since $BA = 0$.

$= tr(A^2) + tr(B^2)$

\#45. $tr(A) = 5 + \lambda_2$ & $\det(A) = 5 \cdot \lambda_2 \Rightarrow 2 = 5 + \lambda_2$ & $-3-4k = 5\lambda_2$

$\Rightarrow \lambda_2 = -3$ So $-4k = -15 + 3 \Rightarrow -4k = -12 \Rightarrow k = 3$ //