HW 9. Solution

Sec. 6.1.

#10. $\det\begin{bmatrix}1&1&1\\1&2&3\\1&3&6\end{bmatrix} = 1\cdot\det(A_{11}) - 1\cdot\det(A_{21}) + 1\cdot\det(A_{31})$

$= 1\cdot\det\begin{bmatrix}2&3\\3&6\end{bmatrix} - 1\cdot\det\begin{bmatrix}1&1\\3&6\end{bmatrix} + 1\cdot\det\begin{bmatrix}1&1\\2&3\end{bmatrix}$

$= 1\cdot(12-9) - 1\cdot(6-3) + 1\cdot(3-2) = 3 - 3 + 1 = 1.$

#18. $\det\begin{bmatrix}0&1&k\\3&2k&5\\9&7&5\end{bmatrix} = -1\cdot\det(A_{12}) + k\cdot\det(A_{13})$

$= -1\cdot\det\begin{bmatrix}3&5\\9&5\end{bmatrix} + k\cdot\begin{bmatrix}3&2k\\9&7\end{bmatrix} = -(15-45)+k(21-18k)$

$= 21k - 18k^2 + 30 = -3(k-2)(6k+5)$.

A is invertible iff $\det(A) \neq 0$ iff $k \neq +2$ & $k \neq -\frac{5}{6}$.

#36. $\det\begin{bmatrix}2&0&2&2\\1&0&2&2\\1&2&2&2\\1&0&1&2\end{bmatrix} = -2\cdot\det(A_{32}) = -2\det\begin{bmatrix}2&2&2\\1&2&2\\1&1&2\end{bmatrix}$

$= -2\left(2\det\begin{bmatrix}2&2\\1&2\end{bmatrix} - 1\cdot\det\begin{bmatrix}2&2\\1&2\end{bmatrix} + 1\cdot\det\begin{bmatrix}2&2\\2&2\end{bmatrix}\right)$

$= -2\left(2\cdot(4-2) - 1\cdot(4-2) + 1\cdot(4-4)\right)$

$= -2(4-2) = -4$

#40. $\det\begin{bmatrix}0&0&3&0&0\\0&0&0&0&2\\0&4&0&0&0\\0&0&0&1&0\\5&0&0&0&0\end{bmatrix} = +5\cdot\det\begin{bmatrix}0&3&0&0\\0&0&0&2\\4&0&0&0\\0&0&1&0\end{bmatrix} = 5\cdot\left(-3\cdot\det\begin{bmatrix}0&0&2\\4&0&0\\0&1&0\end{bmatrix}\right)$

$= -15\cdot\left(2\cdot\det\begin{bmatrix}4&0\\0&1\end{bmatrix}\right) = -15\cdot2\cdot4\cdot1 = -120$

#42. $\det\begin{bmatrix}1&1&0&1&1\\0&2&0&7&0\\2&3&4&5&0\\0&0&0&3&0\\3&4&5&2&6\end{bmatrix} = 3\cdot\det(A_{44}) = 3\cdot\det\begin{bmatrix}1&1&0&1\\0&2&0&0\\2&3&4&0\\3&4&5&6\end{bmatrix} = 3\cdot\left(2\cdot\det\begin{bmatrix}1&0&1\\2&4&0\\3&5&6\end{bmatrix}\right)$

$= 3\cdot2\cdot\left(1\cdot\det\begin{bmatrix}4&0\\5&6\end{bmatrix} + 1\cdot\det\begin{bmatrix}2&4\\3&5\end{bmatrix}\right)$

$= 6\cdot(24+(10-12)) = 6\cdot(24-2) = 6\cdot22 = 132.$

#43. Prove that $\det(-A) = (-1)^n \det(A)$, for any $n\times n$ matrix A.

Let's use induction on the size n of A.

(1) $n=1$: $A=[a] \Rightarrow -A=[-a]$. $\det(-A) = -a = -\det(A)$. So it is true.

(2) Suppose it is true for $(n-1)\times(n-1)$ matrices.

$$\det(-A) = \det\begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & \cdots & \cdots & -a_{nn}\end{bmatrix} = -a_{11}\det(-A_{11}) + a_{12}\det(-A_{12}) - \cdots + (-1)^{n+1}(-a_{1n}\det(-A_{1n}))$$

$(n-1)\times(n-1)$ matrices.

by I.H.

$$= -a_{11}\cdot(-1)^{n-1}\cdot\det(A_{11}) + a_{12}\cdot(-1)^{n-1}\cdot\det(A_{12}) + \cdots + (-1)^{n+1}\cdot(-a_{1n})\cdot(-1)^{n-1}\cdot\det(A_{1n})$$

$$= -(-1)^{n-1}\cdot\left(a_{11}\cdot\det(A_{11}) - a_{12}\cdot\det(A_{12}) + \cdots + (-1)^{n+1}\cdot a_{1n}\det(A_{1n})\right)$$

$$= -(-1)^{n-1}\det(A)$$

$$= (-1)^n\det(A).$$ So it is true.

#44. Prove that $\det(kA) = k^n\cdot\det(A)$, for any $n\times n$ matrix A.

Let's use induction on n.

(1) $n=1$: $A=[a] \Rightarrow kA = [ka]$. So $\det(kA) = ka = k\cdot\det(A)$.

So it is true for $n=1$.

(2) Suppose it is true for $(n-1)\times(n-1)$ matrices.

Let $B = kA$, where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}$.

$(n-1)\times(n-1)$ matrix.

$$\det(B) = \sum_{i=1}^{n}(-1)^{1+i}\cdot b_{1i}\det(B_{1i}) = \sum_{i=1}^{n}(-1)^{1+i}\cdot k a_{1i}\cdot\det(kA_{1i})$$

by I.H.

$$= \sum_{i=1}^{n}(-1)^{1+i} k a_{1i}\cdot k^{n-1}\cdot\det(A_{1i})$$

$$= k\cdot k^{n-1}\cdot\sum_{i=1}^{n}(-1)^{1+i}\cdot a_{1i}\det(A_{1i}) = k^n\cdot\det(A).$$ So it is true.

56. (a). for $n \geq 2$,

$$d_n = \det(M_n) = \det \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 0 \; 0 \end{bmatrix} \quad \swarrow \text{sign } (-1)^{n+1}$$

$$= (-1)^{n+1} \cdot 1 \cdot \det(M_{1n}) \quad \leftarrow \text{Here } M_{1n} = M_{n-1}.$$

$$= (-1)^{n+1} \cdot \det(M_{n-1})$$

$$= (-1)^{n+1} \cdot d_n. \qquad \Rightarrow \boxed{d_n = (-1)^{n+1} d_n, \text{ for all } n \geq 2 \quad (\text{or } (-1)^{n-1} \cdot d_n)}$$

(b)

$d_1 = \det(M_1) = \det[1] = 1.$

$d_2 = \det(M_2) = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$

$d_3 = \det(M_3) = \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1 \cdot (-1) = -1 \quad \left(\text{or } d_3 = (-1)^{3+1} d_2 = -1\right)$

$d_4 = \det(M_4) = \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = d_4 = (-1)^{4+1} d_3 = (-1)^5 \cdot (-1) = 1$

$d_5 = \det(M_5) = (-1)^{5+1} \cdot d_4 = (-1)^6 \cdot 1 = 1$

$d_6 = \det(M_6) = (-1)^{6+1} d_5 = (-1)^7 \cdot 1 = -1$

$d_7 = \det(M_7) = (-1)^{7+1} d_6 = (-1)^8 \cdot (-1) = -1$

$d_8 = \det(M_8) = (-1)^{8+1} d_7 = (-1)^9 (-1) = 1.$

The pattern is $\underbrace{1, -1, -1, 1}, \underbrace{1, -1, -1, 1}, \underbrace{1, -1, -1, 1}, \ldots$

$(1, -1, -1, 1)$ is repeated every four terms.

This means that
$$d_n = \begin{cases} 1 & \text{if the remainder of } n/4 \text{ is } 1 \\ -1 & \text{if } \quad '' \quad n/4 \text{ is } 2 \\ -1 & \text{if } \quad '' \quad n/4 \text{ is } 3 \\ 1 & \text{if } \quad '' \quad n/4 \text{ is } 0. \end{cases}$$

(c) By the pattern, $100 = 4 \cdot 25$. $\Rightarrow$ the remainder when 100 is divided by 4 is 0. $\Rightarrow$ $\underline{d_{100} = 1}$ //

Sec. 6.2

#4. $\begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

$\Rightarrow \det(A) = \det\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix} = 1\cdot 1\cdot 1\cdot 9 = 9$

#8. $\begin{bmatrix} 0&0&0&0&2 \\ 1&0&0&0&3 \\ 0&1&0&0&4 \\ 0&0&1&0&5 \\ 0&0&0&1&6 \end{bmatrix} \xrightarrow{②} \begin{bmatrix} 1&0&0&0&3 \\ 0&0&0&0&2 \\ 0&1&0&0&4 \\ 0&0&1&0&5 \\ 0&0&0&1&6 \end{bmatrix} \xrightarrow{②} \begin{bmatrix} 1&0&0&0&3 \\ 0&1&0&0&4 \\ 0&0&0&0&2 \\ 0&0&1&0&5 \\ 0&0&0&1&6 \end{bmatrix} \xrightarrow{②} \begin{bmatrix} 1&0&0&0&3 \\ 0&1&0&0&4 \\ 0&0&1&0&5 \\ 0&0&0&0&2 \\ 0&0&0&1&6 \end{bmatrix}$

$\xrightarrow{②} \begin{bmatrix} 1&0&0&0&3 \\ 0&1&0&0&4 \\ 0&0&1&0&5 \\ 0&0&0&1&6 \\ 0&0&0&0&2 \end{bmatrix} \Rightarrow \det(A) = (-1)^4 \cdot \det\begin{bmatrix} 1&0&0&0&3 \\ 0&1&0&0&4 \\ 0&0&1&0&5 \\ 0&0&0&1&6 \\ 0&0&0&0&2 \end{bmatrix} = 2.$

#10. $\begin{bmatrix} 1&1&1&1&1 \\ 1&2&3&4&5 \\ 1&3&6&10&15 \\ 1&4&10&20&35 \\ 1&5&15&35&70 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1&1&1&1&1 \\ 0&1&2&3&4 \\ 0&2&5&9&14 \\ 0&3&9&19&34 \\ 0&4&14&34&69 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1&1&1&1&1 \\ 0&1&2&3&4 \\ 0&0&1&3&8 \\ 0&0&3&10&22 \\ 0&0&6&22&53 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1&1&1&1&1 \\ 0&1&2&3&4 \\ 0&0&1&3&8 \\ 0&0&0&1&-2 \\ 0&0&0&4&-13 \end{bmatrix}$

$\xrightarrow{③} \begin{bmatrix} 1&1&1&1&1 \\ 0&1&2&3&4 \\ 0&0&1&3&8 \\ 0&0&0&1&-2 \\ 0&0&0&0&-5 \end{bmatrix} \Rightarrow \det(A) = \det\begin{bmatrix} 1&1&1&1&1 \\ 0&1&2&3&4 \\ 0&0&1&3&8 \\ 0&0&0&1&-2 \\ 0&0&0&0&-5 \end{bmatrix} = 1\cdot 1\cdot 1\cdot 1\cdot(-5) = -5.$

#12. $\vec{v}_1$ & $\vec{v}_4$ are swapped & $\Rightarrow \det\begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_1 \end{bmatrix} = (-1)^1 \cdot \det(A) = -8$

#14. This is the 3rd kind of row operations, so det. doesn't change $\Rightarrow$ det $= 8$.

#16. $\det\begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} \underset{③}{=} \det\begin{bmatrix} (6\vec{v}_1+2\vec{v}_4) - 2(3\vec{v}_1+\vec{v}_4) \\ v_2 \\ v_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} = \det\begin{bmatrix} \vec{0} \\ v_2 \\ v_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} = 0.$

#17. $B = \begin{bmatrix} [T(1)]_{\mathfrak{B}} & [T(t)]_{\mathfrak{B}} & [T(t^2)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 2&3&0 \\ 0&2&6 \\ 0&0&2 \end{bmatrix} \Rightarrow \det(T) = \det(B) = 2\cdot 2\cdot 2 = 8$

#20. $\mathfrak{B} = \{ e_1 = \begin{pmatrix} 1&0 \\ 0&0 \end{pmatrix}, e_2 = \begin{pmatrix} 0&1 \\ 0&0 \end{pmatrix}, e_3 = \begin{pmatrix} 0&0 \\ 1&0 \end{pmatrix}, e_4 = \begin{pmatrix} 0&0 \\ 0&1 \end{pmatrix} \}.$

$B = \begin{bmatrix} [T(e_1)]_{\mathfrak{B}} & [T(e_2)]_{\mathfrak{B}} & [T(e_3)]_{\mathfrak{B}} & [T(e_4)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}$

So $\det B = (-1)\det\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} = -1$

Sec 6.2 (continued)

#38. $\det(A^TA) = \det(A^T)\det(A) = \det(A)\det(A) = 3\cdot 3 = 9$

#40. A: orthogonal $\Leftrightarrow$ $A^TA = I_n$.

$\Rightarrow$ $\det(A^TA) = \det(I_n) = 1$

$\det(A^TA) = (\det(A))^2$ $\Rightarrow$ $\det(A) = \pm 1$

#46. a. $\det\begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix} = 3\cdot\det\begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 3\cdot 7 = 21$

b. $\det\begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = -3\det\begin{bmatrix} b & e \\ c & f \end{bmatrix} + 5\det\begin{bmatrix} a & d \\ c & f \end{bmatrix} - 7\det\begin{bmatrix} a & d \\ b & e \end{bmatrix}$

$= -3(bf-ec) + 5(af-dc) - 7(ae-bd)$.

Since $\det\begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = -(bf-ec) + (af-dc) - (ae-bd) = 7$

& $\det\begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} = -(bf-ec) + 2(af-dc) - 3(ae-bd) = 11$,

we have
$$\left.\begin{aligned} &\left(-(bf-ec) + (af-dc) - (ae-bd)\right) \\ &+ 2\left(-(bf-ec) + 2(af-dc) - 3(ae-bd)\right) \end{aligned}\right\} = 7 + 2\cdot(11) = 29$$

$= -3(bf-ec) + 5(af-dc) - 7(ae-bd) = \det\begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix}$.

$\Rightarrow$ $\det\begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = 29$ //