# HW#7 Solution key

## Sec. 5.3

**#4.** $A$ is orthogonal iff $A^TA = I_n$. for an $n\times n$ matrix.

$$A = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 6 & -3 & 2 \\ 3 & 2 & 6 \end{bmatrix}, \quad A^T = \frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 6 & -3 & 2 \\ -3 & 2 & 6 \end{bmatrix}$$

$$A^TA = \frac{1}{49}\begin{bmatrix} 2 & 6 & 3 \\ 6 & -3 & 2 \\ -3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 6 & -3 \\ 6 & -3 & 2 \\ 3 & 2 & 6 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 4+36+9 & 12-18+6 & -6+12+18 \\ 12-18+6 & 36+9+4 & -18-6+12 \\ -6+12+18 & -18-6+12 & 9+4+36 \end{bmatrix}$$

$$= \frac{1}{49}\begin{bmatrix} 49 & 0 & 24 \\ 0 & 49 & -12 \\ 24 & -12 & 49 \end{bmatrix} \neq I_3.$$ So $A$ is $\boxed{\text{Not}}$ orthogonal. //

**#10.** If $A$ & $B$ are orthogonal, then $B^{-1}$ & $AB$ are orthogonal by Fact 5.3.4.

So $(B^{-1})(AB)$ is also orthogonal. $\Rightarrow \boxed{\text{Yes}}$

**#19.** $B$ is symmetric iff $B^T = B$.

$(-B)^T = -B^T = -B \quad \Rightarrow \quad$ <u>$-B$ is also symmetric.</u>

**#20.** $(AB^2A)^T = A^T(B^2)^T \cdot A^T = A(B^T)^2 A = AB^2A \quad \Rightarrow$ <u>$AB^2A$ is also symmetric</u>

**#24.** $(A^TBA)^T = A^TB^T(A^T)^T = A^TB^TA. \neq A^TBA.$ So <u>it is not symmetric.</u>

**#31.** Since $A^TA = I_n$, $A^T = A^{-1}$. Since $A$ is orthogonal, $A^{-1}$ is also orthogonal. So $A^T$ is orthogonal $\Leftrightarrow$ columns of $A^T$ are orthonormal $\Leftrightarrow$ rows of $A$ are orthonormal. So $\boxed{\text{Yes}}$.

**#40.**

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix} \Rightarrow \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix} - \frac{1}{2} 8 \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{10}\begin{bmatrix} -1 \\ 7 \\ -7 \\ +1 \end{bmatrix} \quad \Rightarrow \quad Q = [\vec{u}_1 \ \vec{u}_2] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{10} \\ \frac{1}{2} & \frac{7}{10} \\ \frac{1}{2} & -\frac{7}{10} \\ \frac{1}{2} & \frac{1}{10} \end{bmatrix}$$

$$\text{Proj}_W = Q \cdot Q^T$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{10} \\ \frac{1}{2} & \frac{7}{10} \\ \frac{1}{2} & -\frac{7}{10} \\ \frac{1}{2} & \frac{1}{10} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -\frac{1}{10} & 7/10 & -\frac{7}{10} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} 26/100 & 18/100 & 32/100 & 24/100 \\ 18/100 & 74/100 & -24/100 & 32/100 \\ 32/100 & -24/100 & 74/100 & 18/100 \\ 24/100 & 32/100 & 18/100 & 26/100 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix} \; //$$

Sec 5.3

#44. Recall that ① $\mathrm{im}(A)^{\perp} = \ker(A^T)$
② $\dim(V) + \dim(V^{\perp}) = k$, where $V \subseteq \mathbb{R}^k$.

Note that $\mathrm{im}(A) \subset \mathbb{R}^n$, since $A$ is an $n\times m$ matrix.

So $\dim(\mathrm{im}(A)) + \dim(\ker A^T) = \dim(\mathrm{im} A) + \dim((\mathrm{im} A)^{\perp}) = n$ //

#45. By rank-nullity Thm, $\underbrace{\dim(\ker A)}_{\dim(\ker A^T)} + \dim(\mathrm{im}(A)) = \underline{\underline{m}}$.

By #44, $n = \dim(\ker A^T) + \dim(\mathrm{im}(A)) = m$.

So for <u>any square</u> matrices $A$, we have <u>$\dim(\ker A) = \dim(\ker A^T)$</u>.

#52. $V = \{ A \in \mathbb{R}^{3\times3} \mid A^T = A \}$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = A^T \iff \begin{matrix} b=d \\ g=c \\ h=f \end{matrix}$$

$$\Rightarrow A = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix} = a\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix} + e\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix} + i\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}$$
$$+ b\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix} + c\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix} + f\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}.$$

We can show that these 6 matrices are linearly independent & span $V$. So $\dim(V) = 6$ & a basis $= \left\{ \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix} \right\}$

Sec. 5.4

#1. $A^T = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, $\ker(A^T)$ = solutions of $A^T(\vec{x}) = \vec{0}$.

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 = -3x_2 \\ x_2 = x_2 \end{cases} \Rightarrow \begin{cases} x_1 = -3/2\,x_2 \\ x_2 = x_2 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} s.$$

A basis of $\ker(A^T) = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$. ("Sketching" is omitted.)

#5. $V = \ker A$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 \end{bmatrix}$. $\Rightarrow A^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 4 \end{bmatrix}$.

$$V^\perp = (\ker A)^\perp = \operatorname{im}(A^T) = \operatorname{Span}\left(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}}_{\text{linearly indep.}}\right) \Rightarrow \text{a basis} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}$$

#7. First, we know that $\operatorname{im}(A)^\perp = \ker(A^T)$.
Since $A$ is symmetric, $A^T = A$. So $\boxed{\operatorname{im}(A)^\perp = \ker(A)} \Leftrightarrow \boxed{\operatorname{im}(A) = (\ker A)^\perp}$

#17. Recall that $\ker(A) = \ker(A^TA) \Rightarrow \operatorname{nullity}(A) = \operatorname{nullity}(A^TA)$.

So Yes: $\operatorname{rank}(A) = m - \operatorname{nullity}(A) = m - \operatorname{nullity}(A^TA) = \operatorname{rank}(A^TA)$.

#20. $$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\vec{b} - A\vec{x}^* = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Since $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$,

$$(\vec{b} - A\vec{x}^*) \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1 + 1 + 0 = 0 \quad \& \quad (\vec{b} - A\vec{x}^*) \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0.$$

So $\underline{\vec{b} - A\vec{x}^* \perp \operatorname{im}(A)}$.

#21. $$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \left(\begin{bmatrix} 6 & 3 & 2 \\ 9 & 8 & 10 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ 3 & 8 \\ 2 & 10 \end{bmatrix}\right)^{-1}\begin{bmatrix} 6 & 3 & 2 \\ 9 & 8 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 49 \\ 0 \end{bmatrix} = \begin{pmatrix} 49 & 98 \\ 98 & 245 \end{pmatrix}^{-1}\begin{bmatrix} 3 \cdot 49 \\ 8 \cdot 49 \end{bmatrix}$$

$$= \frac{1}{49^2}\begin{bmatrix} 245 & -98 \\ -98 & 49 \end{bmatrix}\begin{bmatrix} 3 \cdot 49 \\ 8 \cdot 49 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 49 \cdot 5 & -2 \cdot 49 \\ -2 \cdot 49 & 49 \end{bmatrix}\begin{bmatrix} 3 \cdot 49 \\ 8 \cdot 49 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\|\vec{b} - A\vec{x}^*\| = \sqrt{(\vec{b} - A\vec{x}^*) \cdot (\vec{b} - A\vec{x}^*)} = 42$$

Sec 5.4. [redacted]

5.4. #30.

$$\begin{cases} c_0 = 0 \\ c_0 = 1 \\ c_0 + c_1 = 1 \end{cases} \Rightarrow \begin{array}{c} \overset{A}{\shortparallel} \quad \overset{\vec{b}}{\shortparallel} \\ \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}.$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 = c_0^* \\ 1/2 = c_1^* \end{bmatrix}.$$

So $\underline{\underline{f(t) = \frac{1}{2} + \frac{1}{2}t}}$ approximates the data points.