HW 7. Solution

Sec. 5.1

#2. $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{2^2+3^2+4^2} = \sqrt{29}$.

#6. $\theta = \arccos \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \arccos \dfrac{2-3+8-10}{\sqrt{1+1+4+4}\cdot\sqrt{4+9+16+25}} = \arccos \dfrac{-3}{\sqrt{10}\cdot\sqrt{54}} = \arccos\left(\dfrac{-1}{2\sqrt{15}}\right)$ //

#10. $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u}\cdot\vec{v}=0. \Leftrightarrow 2+3k+4=0 \Leftrightarrow \boxed{k=-2}$.

#12. $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$.

$(\|\vec{v}\|+\|\vec{w}\|)^2 = \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2$.

By Cauchy-Schwartz inequality, $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$ & $\vec{v}\cdot\vec{w} \leq |\vec{v}\cdot\vec{w}|$.

So $\|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\|+\|\vec{w}\|)^2 \underset{\text{both positive}}{\Rightarrow} \|\vec{v}+\vec{w}\| \leq \|\vec{v}\|+\|\vec{w}\|$. ∎

#17. $W^\perp = \{\vec{x}\in\mathbb{R}^n \mid \vec{x}\cdot\vec{v}_1 = 0 \;\&\; \vec{x}\cdot\vec{v}_2=0\}$ where $\vec{v}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}5\\6\\7\\8\end{bmatrix}$.

$= \{\vec{x}\in\mathbb{R}^n \mid \begin{bmatrix}1&2&3&4\\5&6&7&8\end{bmatrix}\vec{x}=0\} = \ker\left(\begin{bmatrix}1&2&3&4\\5&6&7&8\end{bmatrix}\right)$.

$\Rightarrow \begin{bmatrix}1&2&3&4\\5&6&7&8\end{bmatrix} \rightarrow \begin{bmatrix}1&2&3&4\\0&-4&-8&-12\end{bmatrix} \rightarrow \begin{bmatrix}1&2&3&4\\0&1&2&3\end{bmatrix} \rightarrow \begin{bmatrix}1&0&-1&-2\\0&1&2&3\end{bmatrix}$

$$\Rightarrow \begin{cases} x_1 = x_3+2x_4 \\ x_2 = -2x_3-3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = s\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}. \in \operatorname{Span}\left(\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\right).$$

So a basis of $W^\perp = \underline{\left(\begin{bmatrix}1\\-2\\1\\6\end{bmatrix}, \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\right)}$ //

#28. let $\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$. Let $V = \operatorname{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

We can check that $\vec{v}_i\cdot\vec{v}_j = 0$ for $i\neq j$.

So let $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{u}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \dfrac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$, $\vec{u}_3 = \dfrac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$.

Then $\vec{u}_1, \vec{u}_2, \vec{u}_3$ form an orthonormal basis of V.

So $\operatorname{Proj}_V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \left(\vec{u}_1\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\vec{u}_1 + \left(\vec{u}_2\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\vec{u}_2 + \left(\vec{u}_3\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\vec{u}_3$

$= \frac{1}{2}\vec{u}_1 + \frac{1}{2}\vec{u}_2 + \frac{1}{2}\vec{u}_3 = \frac{1}{4}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}3/4\\1/4\\-1/4\\1/4\end{bmatrix}$ //

#29. $\|\vec{x}\|^2 = \sqrt{\vec{x}\cdot\vec{x}}^{\,2} = \|7\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5\|^2 = \|7\vec{u}_1\|^2 + \|-3\vec{u}_2\|^2 + \|2\vec{u}_3\|^2 + \|\vec{u}_4\|^2 + \|-\vec{u}_5\|^2$

by Pythagorean theorems since $\vec{u}_i \perp \vec{u}_j$.

$= 7^2\|\vec{u}_1\|^2 + 3^2\|\vec{u}_2\|^2 + 2^2\|\vec{u}_3\|^2 + \|\vec{u}_4\|^2 + (-1)^2\|\vec{u}_5\|^2$

$= 7^2 + 3^2 + 2^2 + 1 + 1 = 64. \Rightarrow \|\vec{x}\| = \sqrt{64} = \boxed{8}$ //

since $\|\vec{u}_i\| = 1$.

Sec 5.1 & 5.2.

(5.1) #31. Consider $\text{Proj}_V(\vec{x})$ where $V = \text{Span}(\vec{u_1}, \dots, \vec{u_m})$.

Then $\|\text{Proj}_V(\vec{x})\|^2 = p$ (by #29.)

Since $\|\text{proj}_V(\vec{x})\|^2 \leq \|\vec{x}\|^2$, we have $\boxed{p \leq \|\vec{x}\|^2}$ //

Sec 5.2

#6. $\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v_2}^{\perp} = \vec{v_2} - (\vec{u_1}\cdot\vec{v_2})\cdot\vec{u_1} = \begin{bmatrix}3\\4\\0\end{bmatrix} - 3\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\4\\0\end{bmatrix}$.

$\vec{u_2} = \frac{\vec{v_2}^{\perp}}{\|\vec{v_2}^{\perp}\|} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{v_3}^{\perp} = \vec{v_3} - (\vec{u_1}\cdot\vec{v_3})\vec{u_1} - (\vec{u_2}\cdot\vec{v_3})\cdot\vec{u_2}$

$= \begin{bmatrix}5\\6\\7\end{bmatrix} - 5\begin{bmatrix}1\\0\\0\end{bmatrix} - 6\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\7\end{bmatrix}$.

$\vec{u_3} = \frac{\vec{v_3}^{\perp}}{\|\vec{v_3}^{\perp}\|} = \begin{bmatrix}0\\0\\1\end{bmatrix}$. So $\vec{u_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{u_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{u_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are orthonormal vectors.

#7. $\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix}$, $\vec{v_2}^{\perp} = \vec{v_2} - (\vec{u_1}\cdot\vec{v_2})\vec{u_1} = \begin{bmatrix}-2\\1\\2\end{bmatrix} - 0\begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix} = \begin{bmatrix}-2\\1\\2\end{bmatrix}$.

$\vec{u_2} = \frac{\vec{v_2}^{\perp}}{\|\vec{v_2}^{\perp}\|} = \begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}$, $\vec{v_3}^{\perp} = \vec{v_3} - (\vec{u_1}\cdot\vec{v_3})\vec{u_1} - (\vec{u_2}\cdot\vec{v_3})\vec{u_2}$

$= \begin{bmatrix}18\\0\\0\end{bmatrix} - 12\begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix} - (-12)\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix} = \begin{bmatrix}2\\-4\\4\end{bmatrix}$.

$\vec{u_3} = \frac{\vec{v_3}^{\perp}}{\|\vec{v_3}^{\perp}\|} = \begin{bmatrix}1/3\\-2/3\\2/3\end{bmatrix}$ //

#13. Look "back." of the text book solutions.

#19. $M: 3\times2$, $Q: 3\times2$, $R: 2\times2$. Let $\vec{v_1} = \begin{bmatrix}2\\2\\1\end{bmatrix}$, $\vec{v_2} = \begin{bmatrix}1\\1\\5\end{bmatrix}$.

$r_{11} = \|\vec{v_1}\| = 3$, $\vec{u_1} = \frac{\vec{v_1}}{r_{11}} = \begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix}$.

$r_{12} = \vec{u_1}\cdot\vec{v_2} = \frac{2+2+5}{3} = 3$, $\vec{v_2}^{\perp} = \vec{v_2} - r_{12}\vec{u_1} = \begin{bmatrix}1\\1\\5\end{bmatrix} - 3\begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix} = \begin{bmatrix}-1\\-1\\4\end{bmatrix}$

$r_{22} = \|\vec{v_2}^{\perp}\| = 3\sqrt{2}$, $\vec{u_2} = \frac{v_2^{\perp}}{\|\vec{v_2}^{\perp}\|} = \frac{1}{3\sqrt{2}}\begin{bmatrix}-1\\1\\4\end{bmatrix}$.

$\Rightarrow \underline{Q = \begin{bmatrix}\vec{u_1} & \vec{u_2}\end{bmatrix} = \begin{bmatrix}2/3 & -1/3\sqrt{2}\\2/3 & 1/3\sqrt{2}\\1/3 & 4/3\sqrt{2}\end{bmatrix}}$ & $R = \begin{bmatrix}r_{11} & r_{12}\\0 & r_{22}\end{bmatrix} = \underline{\begin{bmatrix}3 & 3\\0 & 3\sqrt{2}\end{bmatrix}}$ //

#21. $M: 3\times3$, $Q: 3\times3$, $R: 3\times3$. Let $\vec{v_1} = \begin{bmatrix}2\\2\\1\end{bmatrix}$, $\vec{v_2} = \begin{bmatrix}-2\\1\\2\end{bmatrix}$, $\vec{v_3} = \begin{bmatrix}18\\0\\0\end{bmatrix}$.

$r_{11} = \|\vec{v_1}\| = 3$, $\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix}$,

$r_{12} = \vec{u_1}\cdot\vec{v_2} = 0$, $\vec{v_2}^{\perp} = \vec{v_2} - r_{12}\vec{u_1} = \vec{v_2} = \begin{bmatrix}-2\\1\\2\end{bmatrix}$.

$r_{22} = \|v_2^{\perp}\| = 3$, $\vec{u_2} = \frac{v_2^{\perp}}{\|\vec{v_2}^{\perp}\|} = \begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}$.

$r_{13} = \vec{u_1}\cdot\vec{v_3} = 12$, $r_{23} = \vec{u_2}\cdot\vec{v_3} = -12$, $\vec{v_3}^{\perp} = \vec{v_3} - r_{13}\vec{u_1} - r_{23}\vec{u_2} = \begin{bmatrix}18\\0\\0\end{bmatrix} - \begin{bmatrix}8\\8\\4\end{bmatrix} + \begin{bmatrix}-8\\4\\8\end{bmatrix} = \begin{bmatrix}2\\-4\\4\end{bmatrix}$.

$r_{33} = \|\vec{v_3}^{\perp}\| = 6$, $\vec{u_3} = \frac{v_3^{\perp}}{\|\vec{v_3}^{\perp}\|} = \begin{bmatrix}1/3\\-2/3\\2/3\end{bmatrix}$.

$\Rightarrow Q = \begin{bmatrix}\vec{u_1} & \vec{u_2} & \vec{u_3}\end{bmatrix} = \underline{\begin{bmatrix}2/3 & -2/3 & 1/3\\2/3 & 1/3 & -2/3\\1/3 & 2/3 & 2/3\end{bmatrix}}$, $R = \begin{bmatrix}r_{11} & r_{12} & r_{13}\\0 & r_{22} & r_{23}\\0 & 0 & r_{33}\end{bmatrix} = \underline{\begin{bmatrix}3 & 0 & 12\\0 & 3 & -12\\0 & 0 & 6\end{bmatrix}}$ //

Sec 5.2

#32. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$. So a basis of the plane is $\left(\underset{\vec{v}_1}{\underset{\parallel}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}}, \underset{\vec{v}_2}{\underset{\parallel}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}} \right)$.

$$\begin{cases} \vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \underline{\dfrac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \quad \vec{v}_2^{\perp} = \vec{V}_2 - (\vec{u}_1 \cdot \vec{v}_2)\,\vec{u}_1 \\ \qquad\qquad = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow \|\vec{v}_2^{\perp}\| = \sqrt{\frac{3}{2}} \\ \vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \dfrac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \dfrac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \underline{\dfrac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}} \,// \end{cases}$$

So an orthonormal basis of V is $\underline{\left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)}$.

#33. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

$$\begin{cases} x_1 = -x_4 \\ x_2 = -x_3 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t \qquad \text{Let } \vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $(\vec{v}_1, \vec{v}_2)$ is a basis of $\ker(A)$.

Since $\vec{v}_1 \perp \vec{v}_2$, $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$ & $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$ form an orthonormal basis of $\ker(A)$.

$$\Rightarrow \underline{\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \,//$$

#41. A : Invertible upper triangular $n \times n$ $\Rightarrow A = \begin{array}{c} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \\ \bigcirc & & & a_{nn} \end{bmatrix} \end{array}$ & $a_{ii} \neq 0$ for all i.

$r_{11} = \left\| \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| = |a_{11}| \Rightarrow \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \pm \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ So let's assume that $a_{11} > 0$. & $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = a_{12} \Rightarrow \vec{v}_2^{\perp} = \vec{v}_2 - r_{12}\vec{u}_1 = \begin{bmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} - a_{12} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow r_{22} = \|\vec{v}_2^{\perp}\| = |a_{22}|. \Rightarrow \vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \pm \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

..... so we can conclude that

$$Q = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} = \begin{bmatrix} (\pm)1 & & \bigcirc \\ & (\pm)1 & \\ & & \ddots & \\ \bigcirc & & & (\pm)1 \end{bmatrix}$$ diagonal with $\begin{cases} \text{positive } 1 & \text{if } a_{ii} > 1 \\ \text{negative } 1 & \text{if } a_{ii} < 1. \end{cases}$

& $R = \begin{bmatrix} |a_{11}| & \pm a_{12} & \cdots & \pm a_{1n} \\ & |a_{22}| & & \vdots \\ & & \ddots & \\ \bigcirc & & & |a_{nn}| \end{bmatrix}$ depending on the sign of Q. //