HW5. Solution

Sec 4.1

#2. ① $p(x)=0$ satisfies $p(2)=0$. So the neutral elt. $p(x)=0$ is in W.

② If $p, q \in W$, then $p(2)=0$ & $q(2)=0$. So $p(2)+q(2)=0$. $\Rightarrow p(x)+q(x) \in W$.

③ If $p \in W$, then $p(2)=0$ & $kp(2)=0$ for all scalar k. So $kp \in W$.

By ①~③, W is a subspace of P_2. Let's find a basis.

If $p(x) \in W$, then $p(2)=0$. Let $p(x)=a_0+a_1x+a_2x^2$. Then $a_0+2a_1+4a_2=0$.

So $a_0 = -2a_1-4a_2$. $\Rightarrow$ $p(x) = (-2a_1-4a_2)+a_1x+a_2x^2$

$= a_1(-2+x)+a_2(-4+x^2)$.

Claim: $\mathcal{B}=(-2+x, -4+x^2)$ forms a basis of W.

① "Span" is done by the above.

② Suppose $c_1(-2+x)+c_2(-4+x^2)=0 \Rightarrow (-2c_1-4c_2)+c_1x+c_2x^2=0$.

Then, since $\{1, x, x^2\}$ is the standard basis for P_2, we must have

$$\begin{cases} -2c_1-4c_2=0 \\ c_1=0 \\ c_2=0 \end{cases} \Rightarrow c_1=0=c_2.$$ Hence $-2+x, -4+x^2$ are linearly independent.

So by ① & ②, $\mathcal{B}$ is a basis of W.

#4. ① $\int_0^1 0\,dt=0 \Rightarrow$ the neutral elt. $p(x)=0$ is in W.

② If $p, q \in W$, then $\int_0^1 p(t)+q(t)\,dt = \int_0^1 p(t)\,dt + \int_0^1 q(t)\,dt = 0+0=0 \Rightarrow p+q \in W$.

③ If $p \in W$, then $\int_0^1 kp(t)\,dt = k\int_0^1 p(t)\,dt = k0 = 0 \Rightarrow kp \in W$.

So by ①~③, W is a subspace of P_2. Let's find a basis.

If $p(x)=a_0+a_1x+a_2x^2 \in W$, then $\int_0^1 a_0+a_1x+a_2x^2\,dx=0$

$\Leftrightarrow \left[a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3\right]_0^1=0 \Leftrightarrow a_0+\frac{a_1}{2}+\frac{a_2}{3}=0. \Rightarrow a_0=-\frac{a_1}{2}-\frac{a_2}{3}$.

So $p(x)=\left(-\frac{a_1}{2}-\frac{a_2}{3}\right)+a_1x+a_2x^2 = a_1\left(-\frac{1}{2}+x\right)+a_2\left(-\frac{1}{3}+x^2\right)$.

Claim: $\mathcal{B}=\left(-\frac{1}{2}+x, -\frac{1}{3}+x^2\right)$ forms a basis of W.

$\longrightarrow$ check: span & linear indep. as we showed in #2.

#6. No! since $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \in W$ & $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix} \in W$, but $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \notin W$.

(or the neutral elt. $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \notin W$).

Check all three properties. $\left(\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow 0 \in W,\ (A+B)\begin{bmatrix}1\\2\\3\end{bmatrix} = A\begin{bmatrix}1\\2\\3\end{bmatrix} + B\begin{bmatrix}1\\2\\3\end{bmatrix} = 0+0=0 \ \cdots \right)$

#10.
$$\begin{bmatrix} a&b&c\\ d&e&f\\ g&h&i \end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow \begin{cases} a+2b+3c=0 \\ d+2e+3f=0 \\ g+2h+3i=0 \end{cases} \Rightarrow \begin{cases} a=-2b-3c \\ d=-2e-3f \\ g=-2h-3i \end{cases}$$

$$\Rightarrow \begin{bmatrix} a&b&c\\ d&e&f\\ g&h&i \end{bmatrix} = \begin{bmatrix} -2b-3c&b&c\\ -2e-3f&e&f\\ -2h-3i&h&i \end{bmatrix} = b\begin{bmatrix} -2&1&0\\0&0&0\\0&0&0 \end{bmatrix} + c\begin{bmatrix} -3&0&1\\0&0&0\\0&0&0 \end{bmatrix} + e\begin{bmatrix} 0&0&0\\-2&1&0\\0&0&0 \end{bmatrix} + f\begin{bmatrix} 0&0&0\\-3&0&1\\0&0&0 \end{bmatrix}$$
$$+ h\begin{bmatrix} 0&0&0\\0&0&0\\-2&1&0 \end{bmatrix} + i\begin{bmatrix} 0&0&0\\0&0&0\\-3&0&1 \end{bmatrix}.$$

So claim: $\mathcal{B} = \left(\begin{bmatrix} -2&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} -3&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\-2&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\-3&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\-2&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\-3&0&1 \end{bmatrix} \right)$ is a basis of W.

#20.
$$A = \begin{bmatrix} a&b\\c&d \end{bmatrix} \ \& -a=+d \Rightarrow A = \begin{bmatrix} a&b\\c&-a \end{bmatrix} = a\begin{bmatrix} 1&0\\0&-1 \end{bmatrix} + b\begin{bmatrix} 0&1\\0&0 \end{bmatrix} + c\begin{bmatrix} 0&0\\1&0 \end{bmatrix}.$$

Claim: $\underline{\mathcal{B} = \left(\begin{bmatrix} 1&0\\0&-1 \end{bmatrix}, \begin{bmatrix} 0&1\\0&0 \end{bmatrix}, \begin{bmatrix} 0&0\\1&0 \end{bmatrix} \right)}$ is a basis of W.

① they span W since we showed in the above.

② if $a\begin{bmatrix} 1&0\\0&-1 \end{bmatrix} + b\begin{bmatrix} 0&1\\0&0 \end{bmatrix} + c\begin{bmatrix} 0&0\\1&0 \end{bmatrix} = \begin{bmatrix} 0&0\\0&0 \end{bmatrix} \Rightarrow \begin{bmatrix} a&b\\c&-a \end{bmatrix} = \begin{bmatrix} 0&0\\0&0 \end{bmatrix}$

So $a=b=c=0$. $\Rightarrow$ they are linearly indep.

Hence it is a basis of W.

& $\underline{\underline{\dim(W) = 3}}$.

#22.
$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \in \mathbb{R}^{n\times n}. \text{ Then } A = a_1\begin{bmatrix} 1&0&\cdots&0\\0&0&\cdots&\\0&&\ddots&\\&\cdots&&0 \end{bmatrix} + a_2\begin{bmatrix} 0&0&\cdots&0\\0&1&0&\\0&&\ddots&\\&&&0 \end{bmatrix} + \cdots + a_n\begin{bmatrix} 0&0&\cdots&0\\ \vdots&&\ddots&0\\0&\cdots&0&1 \end{bmatrix}.$$

Claim: $\mathcal{B} = \left(\begin{bmatrix} 1&0&\cdots&0\\0&&&\vdots\\0&\cdots&&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&\\0&1&0&\\0&&\ddots&\\ \vdots&\cdots&&0 \end{bmatrix}, \cdots, \begin{bmatrix} 0&0&\cdots&0\\ \vdots&&&0\\0&\cdots&0&1 \end{bmatrix} \right)$ is a basis.

Justify this.

So $\underline{\underline{\dim(W) = n}}$.

#24.
$$A = \begin{bmatrix} a&b&c\\0&d&e\\0&0&f \end{bmatrix} = a\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix} + b\begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix} + c\begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix} + d\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix} + e\begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix} + f\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}.$$

Claim: $\mathcal{B} = \left(\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \right)$ is a basis.

So $\underline{\underline{\dim(W) = 6}}$.

#53. Suppose $g_1, \dots, g_m$ are linearly independent in V. Let $\mathcal{B} = (f_1, \dots, f_n)$ be a basis of V. Then we show that $[g_1]_{\mathcal{B}}, \dots, [g_m]_{\mathcal{B}}$ are also linearly indep. in $\mathbb{R}^n$ similarly in Fact 4.1.5. In $\mathbb{R}^n$, there are at most n linearly indep. vectors. So $\underline{\underline{m \le n}}$. So there are at most n linearly indep. elts. in V.

#54. Suppose $\mathcal{B} = (g_1, \dots, g_m)$ is a basis of W which is a subspace of an n-dim'l space V. Then $g_1, \dots, g_m$ are linearly indep. So by #53, $m \le n$ as elts in V.

So this shows that $\dim(W) = m \le n$. ■

Sec 4.2

#2. ① $T(M+N) = 7(M+N) = 7M + 7N = T(M) + T(N)$
② $T(kM) = 7(kM) = 7k\,M = k\,7M = k\,T(M)$. $\Big\} \Rightarrow$ So T is linear.

$\ker(T) = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid T(M) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \Rightarrow 7\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow a=b=c=d=0$.

So $\ker(T) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$. Then since $\ker T = \{0\}$ & $\dim(V) = \dim(\mathbb{R}^{2\times 2}) = 4 = \dim(W)$, by Fact 4.2.4. (c), it is an isomorphism.

#6. ① $T(M+N) = (M+N)\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = M\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + N\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = T(M) + T(N)$
② $T(kM) = (kM)\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = k\left(M\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = k\,T(M)$ $\Big\} \Rightarrow$ So T is linear.

If $\ker(T) \ni M$, $M\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a+3b & 2a+6b \\ c+3d & 2c+6d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} a+3b=0 \\ 2a+6b=0 \end{matrix}$ & $\begin{matrix} c+3d=0 \\ 2c+6d=0 \end{matrix} \Rightarrow \begin{cases} a=-3b \\ c=-3d \end{cases} \Rightarrow \ker(T) = \left\{\begin{pmatrix} -3b & b \\ -3d & d \end{pmatrix}\right\} \neq \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$.

This shows that T is not one-to-one. So it is Not an isom.

#10. ① $T(M+N) = P(M+N)P^{-1} = (PM+PN)P^{-1} = PMP^{-1} + PNP^{-1} = T(M) + T(N)$
② $T(kM) = P(kM)P^{-1} = k\,PMP^{-1} = k\,T(M)$ $\Big\} \Rightarrow$ So T is linear.

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker(T)$, $P\begin{bmatrix} a & b \\ c & d \end{bmatrix}P^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow P^{-1}P\begin{bmatrix} a & b \\ c & d \end{bmatrix}P^{-1}P = P^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}P$.
(since P is invertible, we can do this)

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \ker(T) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$. Since $\dim(V) = \dim(W) = \dim(\mathbb{R}^{2\times 2}) = 4$, & $\ker(T) = \{0\}$, it is an isom.

#20. ① $T((x+iy)+(a+ib)) = T((x+a)+i(y+b)) = x+a-i(y+b)$
$= (x-iy)+(a-ib) = T(x+iy) + T(a+ib)$.
② $T(k(x+iy)) = T(kx+iky) = kx - iky = k(x-iy) = k\,T(x+iy)$ $\Big\} \Rightarrow$ T is linear.

If $x+iy \in \ker(T)$, then $x-iy=0 \Rightarrow x=0$ & $y=0 \Rightarrow x+iy=0$. $\Rightarrow \ker(T)=\{0\} \Rightarrow$
Since $\dim(V) = \dim(W) = \dim(\mathbb{C}) = 2$, it is an isom.

#24. ① $T(f(t)+g(t)) = (f(t)+g(t))''(f(t)+g(t)) = (f''+g'')(f+g) = f''f + f''g + g''f + g''g$

$T(f) + T(g) = f''f + g''g$. $\Rightarrow T(f+g) \neq T(f) + T(g)$.

So T is Not linear. (so don't have to check isom.)

#42. $T(f+g) = \begin{bmatrix} f(7)+g(7) \\ f(11)+g(11) \end{bmatrix} = \begin{bmatrix} f(7) \\ f(11) \end{bmatrix} + \begin{bmatrix} g(7) \\ g(11) \end{bmatrix} = T(f) + T(g)$ $\Big\} \Rightarrow$ T is linear.

$T(kf) = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix} = k\begin{bmatrix} f(7) \\ f(11) \end{bmatrix} = k\,T(f)$.

Since $\dim(P_2) = 3$ & $\dim(\mathbb{R}^2) = 2$, $\dim(P_2) \neq \dim(\mathbb{R}^2)$.
Then by Fact 4.2.4 (b), it can Not be an isom.

Sec 4.2 (Continued)

#52. We showed that #6 is not an isom. Since $\ker(T) = \left\{ \begin{pmatrix} -3b & b \\ -3d & d \end{pmatrix} \right\}$.

$\begin{pmatrix} -3b & b \\ -3d & d \end{pmatrix} = b \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ -3 & 1 \end{pmatrix}$. Since $\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 0 \\ -3 & 1 \end{pmatrix}$ are linearly indep.

involved. and span $\ker(T)$, they form a basis of $\ker(T)$. $\Rightarrow$ $\underline{\text{nullity}(T) = 2}$.

(Then by Rank-nullity theorem, $\dim(\mathbb{R}^{2\times 2}) = \text{rank}(T) + \text{nullity}(T)$
$\quad\quad 4 = \text{rank}(T) + 2$)

$\Rightarrow \underline{\text{rank}(T) = 2}$.

#44. $T(f+g) = \begin{bmatrix} f(1)+g(1) \\ f'(2)+g'(2) \\ f(3)+g(3) \end{bmatrix} = \begin{bmatrix} f(1) \\ f'(2) \\ f(3) \end{bmatrix} + \begin{bmatrix} g(1) \\ g'(2) \\ g(3) \end{bmatrix} = T(f) + T(g)$

$T(kf) = \begin{bmatrix} kf(1) \\ kf'(2) \\ kf(3) \end{bmatrix} = k \begin{bmatrix} f(1) \\ f'(2) \\ f(3) \end{bmatrix} = kT(f)$

$\Rightarrow$ $\underline{T \text{ is linear}}$.

If $f = a_0 + a_1x + a_2x^2 \in \ker(T)$, $\begin{cases} f(1) = a_0 + a_1 + a_2 = 0 \\ f'(2) = a_1 + 4a_2 = 0 \\ f(3) = a_0 + 3a_1 + 9a_2 = 0 \end{cases}$ $\Rightarrow$ Solve: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a_0 = 3a_2 \\ a_1 = -4a_2 \\ a_2 = a_2 \end{array} \Rightarrow f(x) = 3a_2 + (-4a_2)x + a_2x^2$
$= a_2(3 - 4x + x^2)$. for all a_2.

$\Rightarrow \ker(T) = \{ a_2(3-4x+x^2) \mid a_2 \in \mathbb{R} \} \neq \{0\}$.

So T is $\underline{\text{Not an isom}}$.

#60. $T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 + 7a_1 + 49a_2 \\ a_0 + 11a_1 + 121a_2 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 49 \\ 1 & 11 & 121 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

$\Rightarrow \text{im}(T) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \underbrace{\begin{bmatrix} 49 \\ 121 \end{bmatrix}}_{\text{redundant by RREF.}} \right) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \end{bmatrix} \right)$. a basis of $\text{im}(T)$

$\Rightarrow \boxed{\text{rank}(T) = 2}$.

If $a_0 + a_1x + a_2x^2 \in \ker(T)$, then $\begin{cases} a_0 + 7a_1 + 49a_2 = 0 \\ a_0 + 11a_1 + 121a_2 = 0. \end{cases}$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 7 & 49 & 0 \\ 1 & 11 & 121 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 7 & 49 & 0 \\ 0 & 4 & 72 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 7 & 49 & 0 \\ 0 & 1 & 18 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 11 & 0 \\ 0 & 1 & 18 & 0 \end{array}\right]$.

$\Rightarrow \begin{cases} a_0 = -11a_2 \\ a_1 = -18a_2 \end{cases} \Rightarrow \ker(T) = \{ -11a_2 - 18a_2x + a_2x^2 = a_2(-11 - 18x + x^2) \mid a_2 \in \mathbb{R} \}$.

& a basis of $\ker(T) = (-11 - 18x + x^2)$. $\boxed{\text{Span}(-11-18x+x^2)}$

$\Rightarrow \boxed{\text{nullity}(T) = 1}$

Sec 4.2 (Continued)

#74. $Z_n = \{f \in P_n \mid f(0)=0\}. = \{a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in \mathbb{R}\}.$

Define $T: Z_n \longrightarrow P_{n-1}$ by $T(a_1x+a_2x^2+\cdots+a_nx^n) = (a_1x+a_2x^2+\cdots+a_nx^n)'$

① Then T is well-defined, since the derivative of degree $\le n$ has degree $\le n-1$.

② Since $\left.\begin{array}{l} T(f+g) = (f+g)' = f'+g' = T(f)+T(g) \\ \& \ T(kf) = (kf)' = kf' = kT(f) \end{array}\right\}$, T is linear.

③ $\ker(T) = \{f \in Z_n \mid f'=0\}$. So $(a_1x + \cdots + a_nx^n)' = 0 \iff a_1 = a_2 = \cdots = a_n = 0.$

$\Rightarrow f=0. \Rightarrow \ker(T) = \{0\}. \Rightarrow$ T is one-to-one

④ For every $g = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} \in P_{n-1}$, $\left(b_0x + \frac{b_1}{2}x^2 + \cdots + \frac{b_{n-1}}{n}x^n\right)' = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$.

So if $f = b_0x + \frac{b_1}{2}x^2 + \cdots + \frac{b_{n-1}}{n}x^n$, $T(f) = g. \Rightarrow$ T is onto.

By ① ~ ④, T is an isomorphism.

#76. $T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$ by linearity of T.

$\Rightarrow T(0_V) = T(0_V) + T(0_V). \Rightarrow T(0_V) = 0 = 0_W.$ ∎

Sec 4.3

#1. Suppose $af + bg + ch = 0. \Rightarrow (a7 + 9b + 3c) + (3a + 9b + 2c)t + (a + 4b + c)t^2 = 0.$

Since $1, t, t^2$ form a basis of P_2, $\left\{\begin{array}{l} 7a+9b+3c=0 \\ 3a+9b+2c=0 \\ a+4b+c=0 \end{array}\right.$

Solve $\begin{bmatrix} 7 & 9 & 3 \\ 3 & 9 & 2 \\ 1 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & -3 & -1 \\ 0 & -19 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{4}{19} \end{bmatrix} \to [\ \ ] \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow a = b = c = 0$ So they are linearly indep.

#6. $T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \left(T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left(T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

$T\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \left(T\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$

$\Rightarrow \mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ //

Sec 4.3 (continued)

\# 20. P_2 has the standard basis $\mathfrak{B}=\{1, x, x^2\}$.

$T(1) = (1)' = 0 \Rightarrow [T(1)]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$,

$T(x) = x' = 1 \Rightarrow [T(x)]_{\mathfrak{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\Rightarrow$ $\underline{B = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 1\end{bmatrix}}$ //

$T(x^2) = (x^2)' = 2x \Rightarrow [T(x)]_{\mathfrak{B}} = \begin{bmatrix}0\\2\\1\end{bmatrix}$

\#24. $T(1) = 1 \Rightarrow [T(1)]_{\mathfrak{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$T(t-3) = 3-3=0 \Rightarrow [T(t-3)]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\Rightarrow$ $\underline{B = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}}$ //

$T(t-3)^2 = (3-3)^2 = 0 \Rightarrow [T(t-3)^2]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

\# 42. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{\mathfrak{U}} = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)_{\mathfrak{U}} = \begin{bmatrix}1\\0\\1\end{bmatrix}$.

$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}_{\mathfrak{U}} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\Rightarrow$ $\underline{S = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & -1\end{bmatrix}}$.

$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}_{\mathfrak{U}} = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)_{\mathfrak{U}} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

\# 46. $[1]_{\mathfrak{U}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$[t-3]_{\mathfrak{U}} = (-3 + 1\cdot t + 0\cdot t^2)_{\mathfrak{U}} = \begin{bmatrix}-3\\1\\0\end{bmatrix}$ $\Rightarrow$ $\underline{S = \begin{bmatrix}1 & -3 & 9\\0 & 1 & -6\\0 & 0 & 1\end{bmatrix}}$.

$[(t-3)^2]_{\mathfrak{U}} = (t^2 - 6t + 9)_{\mathfrak{U}} = \begin{bmatrix}9\\-6\\1\end{bmatrix}$

\# 48. $\begin{cases} T(\cos(t)) = -\sin(t) \Rightarrow [T(\cos(t))]_{\mathfrak{B}} = \begin{bmatrix}0\\-1\end{bmatrix} \\ T(\sin(t)) = \cos(t) \Rightarrow [T(\sin(t))]_{\mathfrak{B}} = \begin{bmatrix}1\\0\end{bmatrix} \end{cases}$ $\Rightarrow$ $\underline{B = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}}$.

\# 60. a. $\begin{cases} \vec{v}_1 = \vec{u}_1 \Rightarrow [\vec{v}_1]_{\mathfrak{U}} = \begin{bmatrix}1\\0\end{bmatrix}. \\ \vec{v}_2 = \begin{bmatrix}3\\0\\3\end{bmatrix} = \vec{u}_1 + \vec{u}_2 \Rightarrow [\vec{v}_2]_{\mathfrak{U}} = \begin{bmatrix}1\\1\end{bmatrix} \end{cases}$ $\Rightarrow$ $S_{\mathfrak{B}\to\mathfrak{U}} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.

b. $S_{\mathfrak{U}\to\mathfrak{B}} = S^{-1} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$ //

c. $[\vec{u}_1 \ \vec{u}_2] = \begin{bmatrix}1 & 2\\2 & -2\\2 & 1\end{bmatrix}$ & $[\vec{u}_1 \ \vec{u}_2] S_{\mathfrak{B}\to\mathfrak{U}} = \begin{bmatrix}1 & 2\\2 & -2\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 3\\2 & 0\\2 & 3\end{bmatrix}$

$= [\vec{v}_1, \vec{v}_2]$.

So $[\vec{v}_1, \vec{v}_2] = [\vec{u}_1, \vec{u}_2] S_{\mathfrak{B}\to\mathfrak{U}}$. ∎