HW 3 Solution, 2270

Sec. 3.1.

#4. Solve $A\vec{x}=\vec{0}$. $[\underset{}{1}\ \overset{\text{free}}{2\ 3} : 0] \Rightarrow \left[\begin{matrix} x_1 = -2s-3t \\ x_2 = s \\ x_3 = t \end{matrix}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$ & $\ker(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$.

#8. Solve $A\vec{x}=\vec{0}$. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \overset{\text{free}}{-1} & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

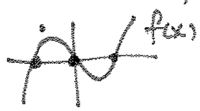
So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right)$. & $\ker(A) = \text{Span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right)$.

#14. $\text{im}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right)$. & Find redundant vectors. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ both columns have leading 1's. So none of them is redundant.

So $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ span $\text{im}(A)$.

#16. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ only leading 1. So $\text{im}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

#27. $f(x) = x^3 - x$. $f(1) = 0 = f(-1)$. So $f^{-1}(0)$ cannot be defined.

But  so $\text{im}(f) = \mathbb{R}$.

#28. $x^2 + \frac{y^2}{4} = 1 \Rightarrow x^2 + \left(\frac{y}{2}\right)^2 = 1. \Rightarrow x = \cos t$ & $\frac{y}{2} = \sin t$.

Let $f: \mathbb{R} \to \mathbb{R}^2$ be defined: $f(t) = \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$. Then $\text{im}(f)$ is the ellipse: $\frac{x^2}{1} + \frac{y^2}{4} = 1$.

#32. $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}_{3\times1}$ For example, $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $T = A = \begin{bmatrix} 7 & 7 & 7 \\ 6 & 6 & 6 \\ 5 & 5 & 5 \end{bmatrix}$.

Then $\text{im}(T) = \text{Span}\left(\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}\right)$.

#40. Since $\ker(A) = \text{im}(B)$, $B\vec{x}$ is in $\ker(A)$ for all $\vec{x}$ in $\mathbb{R}^m$. $\Rightarrow AB\vec{x} = \vec{0}$.

So $\boxed{AB = O_{n\times m}.}$

#50. Yes. Proof: (1) If $\vec{x} \in \ker(A^3)$, $A^3(\vec{x}) = \vec{0} \Rightarrow A(A^3(\vec{x})) = A\vec{0} = \vec{0} \Rightarrow A^4(\vec{x}) = \vec{0}$

$\Rightarrow \vec{x} \in \ker(A^4)$. This shows that $\ker(A^3) \subseteq \ker(A^4)$.

(2) If $\vec{x} \in \ker(A^4)$, $A^4(\vec{x}) = \vec{0}$. Since $A^3(A(\vec{x})) = A^4(\vec{x}) = \vec{0}$,

$A\vec{x} \in \ker(A^3)$. By assumption, since $\ker(A^3) = \ker(A^2)$, $A\vec{x} \in \ker(A^2)$.

So $A^2(A\vec{x}) = \vec{0} \Rightarrow A^3\vec{x} = \vec{0} \Rightarrow \vec{x} \in \ker(A^3)$. $\Rightarrow \ker(A^4) \subseteq \ker(A^3)$.

By (1) & (2), $\ker(A^3) = \ker(A^4)$.

Sec. 3.2

#2. Not a subspace, since if $x \le y \le z$, then $-x \ge -y \ge -z$.

So $-\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is not in W. This violates the 3rd property of a subspace.

#6. a. Yes. let's justify. Since $\vec{0} \in V$ and W, $\vec{0} \in V \cap W$.

If $\vec{v}, \vec{w} \in V \cap W$, then by properties of subspaces of V & W, $\vec{v} + \vec{w} \in V$ and $\vec{v} + \vec{w} \in W \Rightarrow \vec{v} + \vec{w} \in V \cap W$.

If $\vec{v} \in V \cap W$, then $k\vec{v} \in V$ and $k\vec{v} \in W$ for any scalar k. So $k\vec{v} \in V \cap W$.

b. No. A counterexample can be given: In $\mathbb{R}^3$, let $V = \text{Span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $W = \text{Span}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Then $V \cup W$ doesnot contain $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

So $V \cup W$ is not a subspace of $\mathbb{R}^3$.

#8. One way of doing this is: find a and b such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

So need to solve $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 4 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow a = 2$ & $b = -1$.

Then $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} \overset{\text{or}}{\Rightarrow} \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$.

#12. $\begin{bmatrix} 7 & 0 \\ 11 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow rk([\vec{v}_1, \vec{v}_2]) = 1 \neq 2$ So they are <u>not</u> linearly indep. & $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{v}_2$ is redundant.

#15. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow rk([\vec{v}_1, \vec{v}_2, \vec{v}_3]) = 2 \neq 3$.

So they are <u>not</u> linearly indep. and $\vec{v}_3$ is redundant, since $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

#16. $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 6 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rk([\vec{v}_1, \vec{v}_2, \vec{v}_3]) = 2 \neq 3$.

So they are <u>not</u> linearly indep and $\vec{v}_3$ is redundant, since $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

#18. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rk([\vec{v}_1, \vec{v}_2, \vec{v}_3]) = 2 \neq 3$.

So they are <u>not</u> linearly indep & $\vec{v}_3$ is redundant, since $\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

#26. $\begin{array}{c} \qquad\quad \text{redundant} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \to$ So $\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is redundant & $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ← nontrivial relation.

$\Rightarrow -2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \in \ker(A)$.

Sec 3.2 (continued)

#32. $\begin{array}{cccccc} \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

$\begin{array}{cccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array}$

$\text{im}(A) = \text{Span}(v_1, v_2, v_3, v_4, v_5)$.

Here, v_1 & v_3, v_6 are redundant, cause there is no leading 1's in those columns.

So $\text{Span}(v_1, v_2, v_3, v_4, v_5) = \text{Span}(v_2, v_4, v_5)$.

and $[v_2, v_4, v_5]$ has rk 3. So they are linearly indep. Hence $\vec{v_2}, \vec{v_4}, \vec{v_5}$ form a basis of $\text{im}(A)$.

#34. $A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}_{5\times 4}$ & $A\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. $\Rightarrow 1v_1 + 2v_2 + 3v_3 + 4v_4 = 0$.

$\Rightarrow 4v_4 = -v_1 - 2v_2 - 3v_3$

$\Rightarrow \boxed{v_4 = -\frac{1}{4}v_1 - \frac{1}{2}v_2 - \frac{3}{4}v_3}$

#43. Suppose they are dep. Then $\exists$ a, b & c such that

$a\vec{v_1} + b(\vec{v_1} + \vec{v_2}) + c(\vec{v_1} + \vec{v_2} + \vec{v_3}) = 0. \Rightarrow (a+b+c)\vec{v_1} + (b+c)\vec{v_2} + c\vec{v_3} = 0.$

Since $\vec{v_1}, \vec{v_2}$ & $\vec{v_3}$ are indep., $\Rightarrow a+b+c=0,\ b+c=0,\ c=0.$

$\Rightarrow a = 0 = b = c$. So this insists the trivial relation.

Hence $\vec{v_1}$, $\vec{v_1}+\vec{v_2}$, $\vec{v_1}+\vec{v_2}+\vec{v_3}$ are linearly indep.

#45. Let A be an invertible matrix $\Rightarrow$ A must be a square matrix $(n\times n)$ & $\text{rref}(A) = I_n$.

$\Rightarrow \text{rk}(A) = n. \Rightarrow$ all columns are linearly indep.

#48. $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 3x_1 + 4x_2 + 5x_3 = 0 \right\}$.

$= \ker([3\ 4\ 5])$, since solutions of $[3\ \ 4\ \ 5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ are V.

Also, $3x_1 + 4x_2 + 5x_3 = 0.$ implies that x_2 & x_3 can be free. & $x_1 = -\frac{4}{3}x_2 - \frac{5}{3}x_3$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{5}{3} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ s \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} -\frac{4}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}\right).$

$= \text{im}\left(\begin{bmatrix} -\frac{4}{5} & -\frac{5}{3} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$\Rightarrow V = \text{im}\left(\begin{bmatrix} -\frac{4}{5} & -\frac{5}{3} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{im}\left(\begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}\right).$