HW 2 Solution

Section 2.1.

#2. $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 3x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Use Fact 2.1.2.

T can be represented by the matrix $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$. So it is linear.

#4. $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $\Rightarrow A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$. Use Fact 2.1.2.

#6. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+4x_2 \\ 2x_1+5x_2 \\ 3x_1+6x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. So it is linear. & $T = A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

#8. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+7x_2 \\ 3x_1+20x_2 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 3 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$ let $\left[\begin{array}{cc|cc} 1 & 7 & 1 & 0 \\ 3 & 20 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 7 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 7 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 0 & -20 & 7 \\ 0 & 1 & 3 & -1 \end{array}\right]$. So the inverse $T^{-1} = \begin{bmatrix} -20 & 7 \\ 3 & -1 \end{bmatrix}$.

or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -20 & 7 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -20y_1+7y_2 \\ 3y_1 - y_2 \end{bmatrix}$.

(You can use the formular of 2×2 inverse matrices).

#14. a. $\det = 2k-15 \neq 0$ iff A is invertible. So if $k \neq \frac{15}{2}$, it is invertible.

b. By formula, $\det(A)$ must divide 2, 3, 5, & k. to have integer entries. So it must divide the greatest common divisor of 2, 3, 5 & k. First, since the greatest common divisor of 2, 3 & 5 is 1, $\det(A) = 2k-15$ must divide 1 or -1. So it divides k. anyway.

So solve $2k-15 = 1$ & -1 $\Rightarrow$ $\boxed{k=8 \text{ or } k=7}$.

#10. $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}^{-1} = \frac{1}{9-8}\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$ or you can try $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right]$.

#20. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{e}_2$ & $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e}_1$, So it is a reflection about the line $y=x$, since $\vec{e}_1 \xrightarrow{T} \vec{e}_2$ & $\vec{e}_2 \xrightarrow{T} \vec{e}_1$.

#32. A is "scaling" $\Rightarrow$ $A = \begin{bmatrix} 3 & 0 & 0 & \cdots & 0 \\ 0 & 3 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & \cdots & 3 \end{bmatrix}_{n\times n}$.

Section 2.2.

#2. $\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

#6. a unit vector in $L = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$. Use $\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u}$ Recall (Def. 2.2.1)

So $\text{proj}_L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{5}{3}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}$.

#8. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\vec{e}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\vec{e}_2$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\vec{e}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\vec{e}_1$.

So this is a reflection about the line $y=-x$.

#10. a unit vector on $L = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$.

$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\vec{x} = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}\vec{x}$. ← the answer.

#11. $\text{Ref}_L(\vec{x}) = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x} = \begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 1-2u_1^2 \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & -\frac{7}{25} \end{bmatrix}\vec{x}$

#27, #37 Look at the back of the textbook.

#49. $\vec{x}$ is in the unit circle. So $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfies $\underline{x_1^2 + x_2^2 = 1}$.

Then $T(\vec{x}) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 2x_2 \end{bmatrix} \overset{\text{let}}{=} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Look at $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ satisfies "which equation". $y_1 = 5x_1$ & $y_2 = 2x_2$.

$\Rightarrow x_1 = \frac{y_1}{5}$, $x_2 = \frac{y_2}{2}$. $\Rightarrow \left(\frac{y_1}{5}\right)^2 + \left(\frac{y_2}{2}\right)^2 = 1$

So $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is on the ellipse centered at $(0,0)$, $\underline{\underline{\frac{y_1^2}{25} + \frac{y_2^2}{4} = 1}}$.

Section 2.3

$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\1&3&2&0&1&0\\1&0&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&1&-1&1&0\\0&-2&0&-1&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&-1&3&-2&0\\0&1&1&-1&1&0\\0&0&2&-3&2&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&-1&3&-2&0\\0&1&1&-1&1&0\\0&0&1&-\frac{3}{2}&1&\frac{1}{2}\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&\frac{3}{2}&-1&\frac{1}{2}\\0&1&0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&1&-\frac{3}{2}&1&\frac{1}{2}\end{array}\right]$$

3X3 $= A^{-1}$.

#4. $\begin{bmatrix}1&2&1\\1&3&2\\1&0&1\end{bmatrix}\to\begin{bmatrix}1&2&1\\0&1&1\\0&-2&0\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&1\\0&0&2\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=$ rref(A). & square $\Rightarrow$ invertible.

#16. $\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{bmatrix}3&5\\5&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\det(A)=3\cdot 8-5\cdot 5=24-25=-1\neq 0$.
So it is invertible. & inverse $=-1\begin{bmatrix}8&-5\\-5&3\end{bmatrix}=\begin{bmatrix}-8&5\\5&-3\end{bmatrix}$.

#30. $\begin{bmatrix}0&1&b\\-1&0&c\\-b&-c&0\end{bmatrix}\to\begin{bmatrix}1&0&-c\\0&1&b\\-b&-c&0\end{bmatrix}\to\begin{bmatrix}1&0&-c\\0&1&b\\0&-c&-bc\end{bmatrix}\to\begin{bmatrix}1&0&-c\\0&1&b\\0&0&0\end{bmatrix}$ $\Rightarrow$ never invertible.

#34. (a) if $a=0$ or $b=0$ or $c=0$, then rref(A) can't be I_3.
So for any <u>non-zero a, b, c</u>, it is invertible, since its rref is I_3.
& its inverse is $\left[\begin{array}{ccc|ccc}a&0&0&1&0&0\\0&b&0&0&1&0\\0&0&c&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{a}&0&0\\0&1&0&0&\frac{1}{b}&0\\0&0&1&0&0&\frac{1}{c}\end{array}\right]$ $\leftarrow$ inverse.

(b). Similarly, any nonzero diagonal entries make $n\times n$ diagonal matrices invertible.

#38. $\det=-1$. $\Rightarrow A^{-1}=-1\begin{bmatrix}-1&-k\\0&1\end{bmatrix}=\begin{bmatrix}1&k\\0&-1\end{bmatrix}=A$. ("interesting")

Section 2.4

#2. $\begin{bmatrix}1&-1\\-2&2\end{bmatrix}\begin{bmatrix}7&5\\3&1\end{bmatrix}=\begin{bmatrix}4&4\\-8&-8\end{bmatrix}$.

#4. $\begin{bmatrix}1&-1\\0&2\\2&1\end{bmatrix}\begin{bmatrix}3&2\\1&0\end{bmatrix}=\begin{bmatrix}2&2\\2&0\\7&4\end{bmatrix}$.

#13. $[0\;0\;1]\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\vec{e}_3\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}\vec{e}_2=\vec{e}_3\begin{bmatrix}b\\e\\h\end{bmatrix}=h$

#20. False : since $(A-B)(A+B)=AA+A\cdot B-BA-BB=\underline{A^2+AB-BA-B^2}\neq A^2-B^2$.

may not be the same

Any matrices A & B such that $AB\neq BA$ show a counterexample to this statement.

#22. False : cause this implies $AB=BA$ which may not be true.

#23. True : prove it : $(ABA^{-1})^3=(AB\underbrace{A^{-1})(A}_{I_n}BA^{-1})(ABA^{-1})=ABI_nBI_nBA^{-1}=ABBBA^{-1}=AB^3A^{-1}$.

#28. $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}\neq 0_2\Rightarrow A^2=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$.