HW 11. Solution

Sec. 7.4.

#12. $f_A(\lambda) = (2-\lambda)(1-\lambda)^2$, $\lambda = 2$ with a.m = 1, $\lambda = 1$ with a.m = 2.

$E_1 = \ker\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix} = \text{Span}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \Rightarrow$ g.m = 2.

$E_2 = \ker\begin{bmatrix}0&0&1\\0&-1&0\\0&0&-1\end{bmatrix} = \text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \Rightarrow$ g.m = 1.

sum = 3. So it is diagonalizable.

let $S = \begin{bmatrix}-1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ & $D = \begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$. Then $\underline{S^{-1}AS = D}$.

#16. $f_A(\lambda) = (1-\lambda)(\lambda-2)(\lambda-3)$, $\lambda = 1, 2, 3$ with each a.m = 1. $\Rightarrow$ all distinct, so it is diagonalizable.

$E_1 = \ker\begin{bmatrix}3&0&-2\\0&0&0\\1&0&0\end{bmatrix} = \ker\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix} = \text{Span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

$E_2 = \ker\begin{bmatrix}2&0&-2\\0&-1&0\\1&0&-1\end{bmatrix} = \ker\begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix} = \text{Span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$

$E_3 = \ker\begin{bmatrix}1&0&-2\\0&-2&0\\1&0&-2\end{bmatrix} = \ker\begin{bmatrix}1&0&-2\\0&1&0\\0&0&0\end{bmatrix} = \text{Span}\left(\begin{bmatrix}2\\0\\1\end{bmatrix}\right)$

let $S = \begin{bmatrix}0&1&2\\1&0&0\\0&1&1\end{bmatrix}$ & $D = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$. Then $\underline{S^{-1}AS = D}$.

#26. $f_A(\lambda) = (1-\lambda)^2(2-\lambda)$. Since $\lambda = 1$ has a.m = 2, if it has g.m = 2, then A is diagonalizable.

$E_1 = \ker\begin{bmatrix}0&a&b\\0&1&c\\0&0&0\end{bmatrix} = \ker\begin{bmatrix}0&1&c\\0&0&b-ac\\0&0&0\end{bmatrix}$, $\dim E_1 = 2 \iff b-ac=0$.

$\Rightarrow \boxed{b-ac=0}$ //

#32. Find S & D such that $S^{-1}AS = D$.

$f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$. $\Rightarrow \lambda = 2, \lambda = 3$. distinct $\Rightarrow$ diagonalizable.

$E_2 = \ker\begin{bmatrix}2&-2\\1&-1\end{bmatrix} = \ker\begin{bmatrix}1&-1\\0&0\end{bmatrix} = \text{Span}\begin{bmatrix}1\\1\end{bmatrix}$

$E_3 = \ker\begin{bmatrix}1&-2\\1&-2\end{bmatrix} = \ker\begin{bmatrix}1&-2\\0&0\end{bmatrix} = \text{Span}\begin{bmatrix}2\\1\end{bmatrix}$

$\Rightarrow$ let $S = \begin{bmatrix}1&2\\1&1\end{bmatrix}$ & $D = \begin{bmatrix}2&0\\0&3\end{bmatrix}$. Then $S^{-1}AS = D$.

Then $(S^{-1}AS)^t = D^t \Rightarrow S^{-1}A^tS = D^t \Rightarrow A^t = SD^t S^{-1} = \begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}2^t&0\\0&3^t\end{bmatrix}\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$

So $\underline{A^t = \begin{bmatrix}2\cdot3^t - 2^t & 2^{t+1} - 2\cdot3^t\\ -2^t + 3^t & 2^{t+1} - 3^t\end{bmatrix}}$ & $\underline{A^t\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\cdot2^t - 2\cdot3^t\\ 3\cdot2^t - 3^t\end{bmatrix}}$ //

#40. Since C^∞ is infinite-dimensional, we cannot find a matrix for T.

But by the original definition, find λ & $f \neq 0$ such that $T(f) = \lambda f$.

So $T(f) = \lambda f = 5f' - 3f. \iff 5f' - (3+\lambda)f = 0 \leftarrow$ solve the differential equation for f.

$\Rightarrow f_\lambda(x) = c\,e^{\left(\frac{3+\lambda}{5}\right)x}$ for some c. we can let $c=1$. So you can check $T(f_\lambda(x)) = 5f_\lambda' - 3f_\lambda = \lambda f_\lambda$.

So T has an eigenvalue λ for any real number λ & the corresponding eigenfunction $\boxed{f_\lambda(x) = e^{\left(\frac{3+\lambda}{5}\right)x}}$. //

Sec 7.4 (Continued)

#50. Let $\mathfrak{B} = (1, x, x^2)$. Let $A = \mathfrak{B}$-matrix of T.

$$A = \begin{bmatrix} [T(1)]_{\mathfrak{B}} & [T(x)]_{\mathfrak{B}} & [T(x^2)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}. \Rightarrow \lambda = 1 \text{ with a.m} = 3.$$

$$E_1 = \ker \begin{bmatrix} 0 & -3 & 9 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{g.m} = 1 \neq 3.$$

So T is <u>not</u> diagonalizable.

But T has an eigenvector $f(x) = 1 \cdot c$ for some constant $c \neq 0$ with the eigenvalue $\lambda = 1$.

Sec 8.1.

#6. $f_A(\lambda) = -\lambda(\lambda-3)(\lambda+3)$. $\Rightarrow \lambda = 0,\ 3,\ -3$.

$$E_0 = \ker A = \operatorname{Span}\left(\begin{bmatrix}1\\-2\\2\end{bmatrix}\right) \Rightarrow u_1 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$$

$$E_3 = \ker\begin{bmatrix}-3&2&2\\2&-2&0\\2&0&-4\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}2\\2\\1\end{bmatrix}\right) \Rightarrow u_2 = \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}$$

$$E_{-3} = \ker\begin{bmatrix}3&2&2\\2&4&0\\2&0&2\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}-2\\1\\2\end{bmatrix}\right) \Rightarrow u_3 = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix}.$$

these are already perpendicular to each other!

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#10. $f_A(\lambda) = -\lambda^2(\lambda-9)$ $\Rightarrow \lambda = 0, 9$,

$$E_0 = \ker A = \operatorname{Span}\left(\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}\right)$$

$$E_9 = \ker\begin{bmatrix}-8&-2&2\\-2&-5&-4\\2&-4&-5\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\-2\\2\end{bmatrix}\right)$$

$$\Rightarrow u_1 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}.$$

$$u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$$

$$v_3^\perp = v_3 - (u_1 \cdot v_3)u_1 - (u_2 \cdot v_3)u_2$$
$$= \begin{bmatrix}-2\\0\\1\end{bmatrix} - \frac{1}{\sqrt{5}}(-4)\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$$
$$= \begin{bmatrix}-2+\frac{8}{5}\\\frac{4}{5}\\1\end{bmatrix} = \begin{bmatrix}-\frac{2}{5}\\\frac{4}{5}\\1\end{bmatrix}$$
$$\Rightarrow u_3 = \frac{\sqrt{5}}{3}\begin{bmatrix}-\frac{2}{5}\\\frac{4}{5}\\1\end{bmatrix} = \frac{1}{3\sqrt{5}}\begin{bmatrix}-2\\4\\5\end{bmatrix}.$$

$$S = \begin{bmatrix}1/3 & 2/\sqrt{5} & -2/3\sqrt{5}\\-2/3 & 1/\sqrt{5} & 4/3\sqrt{5}\\2/3 & 0 & 5/3\sqrt{5}\end{bmatrix}, \quad D = \begin{bmatrix}9&0&0\\0&0&0\\0&0&0\end{bmatrix}. \Rightarrow \boxed{A = SDS^T}.$$

#24. $f_A(\lambda) = (\lambda+1)^2(\lambda-1)^2$, $\lambda = 1, -1$.

$$E_1 = \ker\begin{bmatrix}-1&0&0&1\\0&-1&1&0\\0&1&-1&0\\1&0&0&-1\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)$$

$$E_{-1} = \ker\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}\right)$$

$$\Rightarrow \begin{cases} u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \\ u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} \\ u_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \\ u_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}. \end{cases}$$

#31. Since A is symmetric, we can find S & D such that $A = SDS^{-1}$.

$A^2 = SD^2S^{-1}$. $\quad rk(A) = rk(D)$ & $rk(A^2) = rk(D^2) = rk(D) = rk(A)$

(Since it depends only on diagonal entries.)

So "<u>True</u>"

Sec. 8.3

\#4. $f_A(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$, $\Rightarrow \lambda = 1, 1 \Rightarrow \underline{\sigma_1 = 1 \geq \sigma_2 = 1.}$

\#6. $f_A(\lambda) = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$. $\Rightarrow \lambda = 5, 0 \Rightarrow \sigma_1 = \sqrt{5} \geq \sigma_2 = 0$.

$E_5 = \ker\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = \ker\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \text{Span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\Rightarrow \left\{ \begin{array}{l} v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}. \\ v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{array} \right\}$ forms an orthonormal basis of $\mathbb{R}^2$.

$E_0 = \ker\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \ker\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \text{Span}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\Rightarrow$

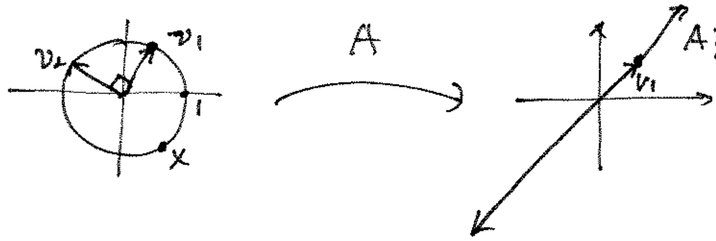
Then $\|A\vec{v}_1\| = \|5\vec{v}_1\| = \sqrt{5}$ & $\|\vec{v}_1\| = 1$.

Note that $A\vec{v}_2 = \vec{0}$. $\Rightarrow \|A\vec{v}_2\| = 0$.

Let $\vec{x}$ be on the unit circle. Then $\vec{x} = \cos t \cdot v_1 + \sin t\, \vec{v}_2$, for t in $\mathbb{R}$.

$\Rightarrow$ $\underline{A\vec{x} = 5\cos t \cdot v_1}$ so the image $A\vec{x}$ is always in the line spanned by v_1.

So



$A\vec{x}$: a $\underline{\text{straight line!}}$

<u>Sec 8.3 (Continued)</u>

#11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}_{3\times 2}$, $\quad A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

$\Rightarrow \lambda = 4, 1. \quad \sigma_1 = 2, \ \sigma_2 = 1.$

$E_4 = \ker\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, $\quad E_1 = \ker\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

$\Rightarrow \ v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $Av_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad Av_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

let $u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ & $U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Then $\underline{\underline{A = U\Sigma V^T}}$.

#12. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2}$, $\quad A^TA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow f(\lambda) = \lambda^2 - 2\lambda$.

$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = 0 \Rightarrow \sigma_1 = \sqrt{3}, \lambda_2 = 0$

$E_3 = \ker\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \ker\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$E_0 = \ker\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$\Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$Av_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad Av_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$\Rightarrow \ u_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad u_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

let $U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$, $\quad V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

& $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A = U\Sigma V^T$.

#14. $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}_{2\times 2}$. $\quad A^TA = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} \Rightarrow f_{A^TA}(\lambda) = \lambda^2 - 17\lambda + 16$

$\Rightarrow \lambda = 16, \ 1, \quad \sigma_1 = 4, \ \sigma_2 = 1.$

$E_{16} = \ker\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} = \ker\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \Rightarrow v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$E_1 = \ker\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} = \ker\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$

$Av_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad Av_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad \Rightarrow \quad u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{2} \\ 1/\sqrt{5} & 1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A = U\Sigma V^T.$ //