HW #10 Solution key

Sec 7.3.

#4. $A=\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}$ $\quad f_A(\lambda) = \lambda^2-2\lambda+1 = (\lambda-1)^2 \Rightarrow \lambda = 1$ with a.m = 2.

$E_1 = \ker\begin{bmatrix}0-1 & -1\\ 1 & 2-1\end{bmatrix} = \ker\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix} = \ker\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$ ← basis.

& $\underline{g.m.(\lambda=1) = 1} \Rightarrow$ So A has <u>No</u> eigenbasis.

#8. $f_A(\lambda) = \det\begin{bmatrix}1-\lambda & 1 & 0\\ 0 & 2-\lambda & 2\\ 0 & 0 & 3-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda)(3-\lambda) \Rightarrow \lambda = 1, 2, 3$ with each a.m = 1.

$E_1 = \ker\begin{bmatrix}0 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix} = \ker\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 2\end{bmatrix} = \ker\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ ← basis for $\lambda=1$.

$E_2 = \ker\begin{bmatrix}-1 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 1\end{bmatrix} = \ker\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$ ← basis for $\lambda=2$.

$E_3 = \ker\begin{bmatrix}-2 & 1 & 0\\ 0 & -1 & 2\\ 0 & 0 & 0\end{bmatrix} = \ker\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix} = \ker\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)$ ← basis for $\lambda=3$.

Since A has 3 distinct eigenvalues, A has an eigenbasis $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right)$.

#14. $f_A(\lambda) = \det\begin{bmatrix}1-\lambda & 0 & 0\\ -5 & -\lambda & 2\\ 0 & 0 & 1-\lambda\end{bmatrix} = (1-\lambda)\det\begin{bmatrix}-\lambda & 2\\ 0 & 1-\lambda\end{bmatrix} = (1-\lambda)(-\lambda)(1-\lambda) = \underline{-\lambda(1-\lambda)^2}$

$\Rightarrow \lambda = 0$ with a.m = 1 & $\lambda = 1$ with a.m = 2.

$E_0 = \det\begin{bmatrix}1 & 0 & 0\\ -5 & 0 & 2\\ 0 & 0 & 1\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 0 & 1\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ ← basis for $\lambda=0$. $\Rightarrow g.m(0) = 1$

$E_1 = \det\begin{bmatrix}0 & 0 & 0\\ -5 & -1 & 2\\ 0 & 0 & 0\end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix}1\\-5\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}\right)$ ← basis for $\lambda=1$. $\Rightarrow g.m(1) = 2$

$1+2 = 3 = \dim(\mathbb{R}^3)$. $\Rightarrow A$ has an eigenbasis $\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-5\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}\right)$.

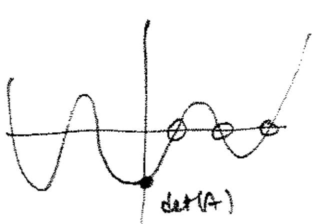
#21. $\begin{bmatrix}1\\2\end{bmatrix} \in \ker\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix} - 1\cdot I_3\right)$ & $\begin{bmatrix}2\\3\end{bmatrix} \in \ker\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix} - 2\cdot I_3\right)$

$\Rightarrow \begin{pmatrix}a-1 & b\\ c & d-1\end{pmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ & $\begin{bmatrix}a-2 & b\\ c & d-2\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\Rightarrow a = 5,\ b = -2,\ c = 6,\ d = -2.$

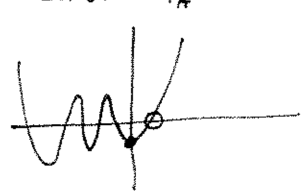
So $A = \begin{bmatrix}5 & -2\\ 6 & -2\end{bmatrix}$ //

#26. $f_A(\lambda) = (-\lambda)^6 + tr(A)(-\lambda)^5 + \cdots + \det(A) = \lambda^6 - tr(A)\lambda^5 + \cdots + \det(A)$ ← negative.

& $\det(A) = f_A(0)$. i.e. this is the y-intercept of $y = f_A(\lambda)$.



or



So $f_A(\lambda)$ has a root on the positive x-axis.

So A has at least one positive eigenvalue.

Sec 7.3 (continued)

#27. $f_A(\lambda) = (-\lambda)^2 + tr(A)(-\lambda) + \det(A) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$.

So $\lambda = 2$ & 3. with a.m $= 1$ each.

#28. $f_{J_n(k)}(\lambda) = \det \begin{bmatrix} k-\lambda & 1 & 0 & \cdots & 0 \\ 0 & k-\lambda & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & k-\lambda \end{bmatrix} = (k-\lambda)^n$.

$\Rightarrow \lambda = k$ with a.m $= n$.

$E_k = \ker(J_n(k) - k I_n) = \ker \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & & 1 \\ \cdots & & & & 0 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \text{Span}(\vec{e}_1)$.

So $\lambda = k$ has g.m $= 1$. (So $J_n(k)$ has <u>no</u> eigenbasis.)

#36. No! since they have different traces.