HW1. Solution

Sec 1.1.

#6. $\left|\begin{array}{l} x+2y+3z=8 \\ x+3y+3z=10 \\ x+2y+4z=9 \end{array}\right| \xrightarrow[-①+③\Rightarrow③]{-①+②\Rightarrow②} \left|\begin{array}{l} x+2y+3z=8 \\ y=2 \\ z=1 \end{array}\right| \xrightarrow[-3③+①\Rightarrow①]{-2②+①\Rightarrow①} \left|\begin{array}{l} x=1 \\ y=2 \\ z=1 \end{array}\right|$ $\Rightarrow$ $\boxed{\text{unique solution } x=1,\ y=2,\ z=1.}$

#8. $\left|\begin{array}{l} x+2y+3z=0 \\ 4x+5y+6z=0 \\ 7x+8y+10z=0 \end{array}\right| \xrightarrow[-7①+③\Rightarrow③]{-4①+②\Rightarrow②} \left|\begin{array}{l} x+2y+3z=0 \\ -3y-6z=0 \\ -6y-11z=0 \end{array}\right| \xrightarrow{\frac{②}{-3}\Rightarrow②} \left|\begin{array}{l} x+2y+3z=0 \\ y+2z=0 \\ -6y-11z=0 \end{array}\right| \xrightarrow[6②+③\Rightarrow③]{-2②+①\Rightarrow①} \left|\begin{array}{l} x \quad -z=0 \\ y+2z=0 \\ z=0 \end{array}\right|$

$\xrightarrow[4③+①\Rightarrow①]{-2③+②\Rightarrow②} \left|\begin{array}{l} x=0 \\ y=0 \\ z=0 \end{array}\right|$ $\Rightarrow$ $\boxed{\text{unique solution } x=0,\ y=0,\ z=0.}$

#26. $\left|\begin{array}{l} x+y-z=2 \\ x+2y+z=3 \\ x+y+(k^2-5)z=k \end{array}\right| \xrightarrow[-①+③\Rightarrow③]{-①+②\Rightarrow②} \left|\begin{array}{l} x+y-z=2 \\ y+2z=1 \\ (k^2-4)z=k-2 \end{array}\right|$, Look at $(k^2-4)z=k-2$.
$\Rightarrow (k+2)(k-2)z=(k-2)$.
So we have three cases, $k=2$ or $k=-2$ or $k\neq 2,-2$

If $k=2$, then this becomes $0=0$. $\Rightarrow \left|\begin{array}{l} x+y-z=2 \\ y+2z=1 \\ 0=0 \end{array}\right| \xrightarrow{-②+①\Rightarrow①} \left|\begin{array}{l} x \quad -3z=1 \\ y+2z=1 \end{array}\right|$

So if $z=t$, (free variable), then $\boxed{\begin{cases} x=3t+1, \\ y=-2t+1, \\ z=t \end{cases} \text{for arbitrary } t.}$ (infinitely many solutions)

If $k=-2$, then $(k+2)(k-2)z=k-2 \Rightarrow 0=-4$ ✗ impossible. $\Rightarrow$ $\boxed{\text{No solution}}$

If $k\neq -2, 2$, then $(k+2)(k-2)\neq 0$. So divide by $(k+2)(k-2)$. $\Rightarrow$ Last row becomes $z=\frac{1}{k+2}$.

So $\left|\begin{array}{l} x+y-z=2 \\ y+2z=1 \\ z=\frac{1}{k+2} \end{array}\right| \xrightarrow[\substack{3③+①\Rightarrow① \\ -2③+②\Rightarrow②}]{-②+①\Rightarrow①} \left|\begin{array}{l} x=\frac{k+5}{k+2} \\ y=\frac{k}{k+2} \\ z=\frac{1}{k+2} \end{array}\right|$ $\Rightarrow$ $\boxed{\text{unique solution, } x=\frac{k+5}{k+2},\ y=\frac{k}{k+2},\ z=\frac{1}{k+2}.}$

#30. $(1,p) \Rightarrow a+b+c=p$
$(2,q) \Rightarrow a+2b+4c=q$
$(3,r) \Rightarrow a+3b+9c=r$
$\Rightarrow$ solve this linear system for a, b & c.

$\Rightarrow \left|\begin{array}{l} a=p+q-r \\ b=\frac{-p+4q-3r}{-2} \\ c=\frac{r+p-2q}{-2} \end{array}\right|$ $\Rightarrow$ So such a polynomial exists and it is

$\boxed{f(t)=(p+q-r)+\left(\frac{-p+4q-3r}{-2}\right)t+\left(\frac{r+p-2q}{-2}\right)t^2}$

$\Downarrow$
unique solution.

Sec 1.2.

#6. $\begin{bmatrix} 1 & -7 & 0 & 0 & 1 & \vdots & 3 \\ 0 & 0 & 1 & 0 & -2 & \vdots & 2 \\ 0 & 0 & 0 & 1 & 1 & \vdots & 1 \end{bmatrix}$ this is already in rref with 2 free variables x_2, x_5.

Let $x_2 = s$, $x_5 = t$ for arbitrary s & t.

Then $\begin{cases} x_1 = 7s - t + 3 \\ x_2 = s \\ x_3 = 2t + 2 \\ x_4 = -t + 1 \\ x_5 = t \end{cases}$ for any s & t.

(infinitely many solutions).

#18. (a) No, since 1 on (3,3) entry can be eliminated by using row ②.

(b) Yes.

(c) No, since the 3rd row has a leading 1 but the row above (2nd row) doesn't have a leading 1.

(d) Yes.

#22. For some a & b, there are 6 types:

$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$, $\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

#30. $\left.\begin{array}{l} (0,1) \Rightarrow a + 0\cdot b + 0\cdot c + 0\cdot d = 1 \\ (1,0) \Rightarrow a + b + c + d = 0 \\ (-1,0) \Rightarrow a - b + c - d = 0 \\ (2,-15) \Rightarrow a + 2b + 4c + 8d = -15 \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 1 & 1 & 1 & 1 & \vdots & 0 \\ 1 & -1 & 1 & -1 & \vdots & 0 \\ 1 & 2 & 4 & 8 & \vdots & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 1 & 1 & \vdots & -1 \\ 0 & -1 & 1 & -1 & \vdots & -1 \\ 0 & 2 & 4 & 8 & \vdots & -16 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 1 & 1 & \vdots & -1 \\ 0 & 0 & 2 & 0 & \vdots & -2 \\ 0 & 0 & 2 & 6 & \vdots & -14 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 0 & \frac{6}{6} & \vdots & \frac{-12}{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 0 & 1 & \vdots & -2 \end{bmatrix} \Rightarrow a = 1,\ b = 2,\ c = -1,\ d = -2.$

$\Rightarrow \boxed{f(t) = 1 + 2t - t^2 - 2t^3}$

#35. Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix} = 0$.

$\Rightarrow$ get the system $\begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 0 \\ 1 & 2 & 3 & 4 & \vdots & 0 \\ 1 & 9 & 9 & 7 & \vdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 0 \\ 0 & 1 & 2 & 3 & \vdots & 0 \\ 0 & 8 & 8 & 6 & \vdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 & \vdots & 0 \\ 0 & 1 & 2 & 3 & \vdots & 0 \\ 0 & 0 & \frac{-8}{-8} & \frac{-18}{-8} & \vdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & \vdots & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \vdots & 0 \\ 0 & 0 & 1 & \frac{9}{4} & \vdots & 0 \end{bmatrix}$

For free variable $x_4 = t$, $x_1 = -\frac{1}{4}t$, $x_2 = \frac{3}{2}t$, $x_3 = -\frac{9}{4}t$, $x_4 = t$.

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}t \\ \frac{3}{2}t \\ -\frac{9}{4}t \\ t \end{bmatrix} \left(= t \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{2} \\ -\frac{9}{4} \\ 1 \end{bmatrix} \text{ for any } t \right)$ or $\left(= s \begin{bmatrix} -1 \\ 6 \\ -9 \\ 4 \end{bmatrix} \text{ for any } s \right)$.

#36. get the system $\begin{cases} x_1 + 2x_2 + 4x_3 = -8 \\ 4x_1 + 5x_2 + 6x_3 = -1 \\ 7x_1 + 8x_2 + 9x_3 = 2 \\ 5x_1 + 3x_2 + x_3 = 15 \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 & 4 & \vdots & -8 \\ 4 & 5 & 6 & \vdots & -1 \\ 7 & 8 & 9 & \vdots & 2 \\ 5 & 3 & 1 & \vdots & 15 \end{bmatrix} \to \cdots \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 3 \\ 0 & 0 & 1 & \vdots & -4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

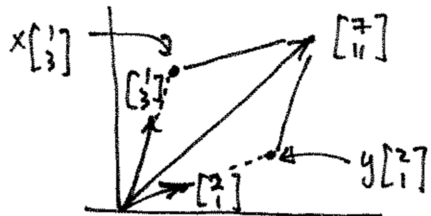
So we have a unique solution $x_1 = 2$, $x_2 = 3$, $x_3 = -4$.

So $\underline{\vec{b} = 2\vec{v}_1 + 3\vec{v}_2 - 4\vec{v}_3}$ as a linear combination!

Sec 1.3.

#5. (a) $\begin{bmatrix}1&2\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}7\\11\end{bmatrix}$. $\Rightarrow$ $x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}7\\11\end{bmatrix}$.

(b) So $\begin{bmatrix}7\\11\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\end{bmatrix}$ & $\begin{bmatrix}2\\1\end{bmatrix}$.



$\left[\begin{array}{cc|c}1&2&7\\3&1&11\end{array}\right]\to\left[\begin{array}{cc|c}1&0&3\\0&1&2\end{array}\right]$.

$\Rightarrow x=3$ & $y=2$.

$\Rightarrow$ $\begin{bmatrix}7\\11\end{bmatrix}$ is the sum of $3\begin{bmatrix}1\\3\end{bmatrix}$ & $2\begin{bmatrix}2\\1\end{bmatrix}$.

by columns

#14. ① $\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}\begin{bmatrix}-1\\2\\1\end{bmatrix}=-1\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}2\\3\end{bmatrix}+1\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-1+4+3\\-2+6+4\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$ //

by row

② $\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}\begin{bmatrix}-1\\2\\1\end{bmatrix}=\begin{bmatrix}[1\,2\,3]\cdot[-1\,2\,1]\\ [2\,3\,4]\cdot[-1\,2\,1]\end{bmatrix}=\begin{bmatrix}-1+4+3\\-2+6+4\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$ // same!

#18. $\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}1\\3\\5\end{bmatrix}+2\begin{bmatrix}2\\4\\6\end{bmatrix}=\begin{bmatrix}1+4\\3+8\\5+12\end{bmatrix}=\begin{bmatrix}5\\11\\17\end{bmatrix}$ //

#20. a. $\begin{bmatrix}9&8\\7&6\\6&6\end{bmatrix}$. b. $\begin{bmatrix}9&-9&18\\27&36&45\end{bmatrix}$ //

#30. Rank 1 - 3x3 matrix can look like $\begin{bmatrix}a&b&c\\ka&kb&kc\\la&lb&lc\end{bmatrix}$ $\left(\to\begin{bmatrix}a&b&c\\0&0&0\\0&0&0\end{bmatrix}\right)$

So find a, b, c such that $\begin{bmatrix}a&b&c\\ka&kb&kc\\la&lb&lc\end{bmatrix}\begin{bmatrix}5\\3\\-9\end{bmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$.

For example, $\begin{bmatrix}\frac{5}{2}&0&0\\0&0&0\\-\frac{1}{4}&0&0\end{bmatrix}$ works!

#34. a. $A\vec{e}_1$ = 1st column of $A=\begin{bmatrix}a\\d\\g\end{bmatrix}$,

$A\vec{e}_2$ = 2nd column of $A=\begin{bmatrix}b\\e\\h\end{bmatrix}$,

$A\vec{e}_3$ = 3rd column of $A=\begin{bmatrix}c\\f\\k\end{bmatrix}$

We did this in class.

b. $B\vec{e}_1=\vec{v}_1$, $B\vec{e}_2=\vec{v}_2$, $B\vec{e}_3=\vec{v}_3$. //