Name : Solution

Student ID # :

Math 2270 - 2
Spring 2006
Instructor: Bo-Hae Im

EXAM 3
Thursday, April 11, 2006

Problem	points	score
1	8	
2	10	
3	5	
4	7	
5	15	
Total	45	

1.(8 pts) Let M_n be the $n \times n$ matrix whose column vectors consist of $\vec{e}_n, \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{n-1}$ in order. For example,

$$M_1 = \begin{bmatrix}1\end{bmatrix}, M_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, M_3 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}, M_4 = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}.$$

Let $d_n = \det(M_n)$.

(1) Find d_1, d_2, d_3, d_4. Do you see a pattern?

$d_1 = 1$

$d_2 = -1$

$d_3 = 1$

$d_4 = -1$

Pattern is :

$d_n = (-1)^{n+1} \det(I_{n-1})$

$= (-1)^{n+1} \cdot 1$

(2) Find a closed formula for d_n for $n \geq 1$ and find d_{150}.

$d_n = (-1)^{n+1}$ or $(-1)^{n-1}$

$d_{150} = (-1)^{151} = -1$

2.(each 2 pts) Decide which of the following statements are true. (You don't have to justify your answer. Just answer 'T' for 'True' and 'F' for 'False' statement. No partial credit will be given.)

(1) If A is invertible, then $adj(A)$ is also invertible. T

(2) If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of A associated to the SAME eigenvalue λ, then $\vec{v}+\vec{w}$ is also an eigenvector of A associated to λ. T

(3) If the determinant of a 3×3 matrix A is 3, then its rank must be 3. T

(4) There exists a 5×5 matrix A such that $A^2 = -I_5$. F

(5) If an invertible matrix A has an eigenvalue λ, then A^{-1} must have λ as an eigenvalue. F

3.(5 pts) Let $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}$ be the 3×3 matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Suppose $\det(A) = 3$. Find $\det\left(\begin{bmatrix} -3\vec{v}_2 + 2\vec{v}_1 \\ -2\vec{v}_3 \\ \vec{v}_1 \end{bmatrix}\right)$. (Please explain which row operations are applied to obtain this matrix from the original A to get partial credit.)

$$\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix} \xrightarrow{\text{Swap}} \begin{bmatrix} \vec{v}_3 \\ \vec{v}_2 \\ \vec{v}_1 \end{bmatrix} \xrightarrow{\text{Swap}} \begin{bmatrix} \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix} \xrightarrow[\cdot(-2)]{\cdot(-3)} \begin{bmatrix} -3\vec{v}_2 \\ -2\vec{v}_3 \\ \vec{v}_1 \end{bmatrix} \xrightarrow[\Rightarrow 1]{2\cdot 3 + 1} \begin{bmatrix} -3\vec{v}_2 + 2\vec{v}_1 \\ -2\vec{v}_3 \\ \vec{v}_1 \end{bmatrix}$$

(doesn't effect "det")

So $$\det\begin{bmatrix} -3\vec{v}_2 + 2\vec{v}_1 \\ -2\vec{v}_3 \\ \vec{v}_1 \end{bmatrix} = (-1)^2(-3)(-2)\det\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= 6 \cdot 3 = \boxed{18}$$

4.(7 pts) Consider the parallelepiped V in $\mathbb{R}^3$ defined by three vectors

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Find the volume of the image $T(V)$ of V under the linear transformation $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 5 & 6 \\ 3 & 2 & 0 \end{bmatrix} \vec{x}.$

$$\text{Vol}(T(V)) = |\det(V)| \cdot |\det(T)|$$

$$= |(-1)(-2)(-2)| \cdot \left| (-6) \cdot \det \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \right|$$

$$= 4 \cdot 6$$

$$= 24$$

5. (15 pts) Consider the linear transformation $T : P_2 \to P_2$ defined by $T(f(x)) = f(2x+1)$.
   (1) Find a matrix A of T with respect to the standard basis of P_2.
   (2) Find all eigenvalues of T with specifying the algebraic multiplicity
   (3) Find a basis of each eigenspace of P_2 with respect to T with specifying geometric multiplicity for each eigenvalue.
   (4) Determine and explain whether there exists an eigenbasis of P_2 for T or not.

(1) $A = \begin{bmatrix} [T(1)]_{\mathcal{B}} & [T(x)]_{\mathcal{B}} & [T(x^2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

$(\det(T) = \det(A) = 1 \cdot 2 \cdot 4 = 8)$ not necessary!

(2) $f_A(\lambda) = (1-\lambda)(2-\lambda)(4-\lambda)$

So eigenvalues are 1, 2 & 4 with a.m 1 each.

(3) $E_1 = \ker(A - 1 \cdot I_3) = \ker \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$E_2 = \ker(A - 2 \cdot I_3) = \ker \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

$E_4 = \ker(A - 4 \cdot I_3) = \ker \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$

P_2 has a basis 1 for eigenvalue 1, with g.m $= 1$
a basis $x+1$ for eigenvalue 2, with g.m $= 1$
and a basis x^2+2x+1 for eigenvalue 4 with g.m $= 1$.

(4) The sum of geometric multiplicities is 3 which is the dimension of P_2,

So P_2 has an eigenbasis consisting of $(1, x+1, x^2+2x+1)$.