

Name : Solution

Student ID # : \_\_\_\_\_

Math 2270 - 2  
Spring 2006  
Instructor: Bo-Hae Im

**EXAM 2**  
**Thursday, March 9, 2006**

Problem	points	score
1	6	
2	11	
3	6	
4	8	
5	7	
6	7	
EC	5	
Total	45(+5)	

1.(6 pts) Find a basis for the space of all upper triangular  $2 \times 2$  matrices and find the dimension of the space. (Explain why the elements you find for a basis really form a basis for this space.)

$$\begin{aligned} \text{let } V &= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \text{ are in } \mathbb{R} \right\} \\ &= \left\{ 2 \times 2 \text{ upper triangular matrices} \right\}. \end{aligned}$$

$$\text{Then } \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ is } \underline{\text{spanned}} \text{ by } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ \& } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\text{Suppose } a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Leftrightarrow a=0, \quad b=0, \quad c=0.$$

$$\text{So } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ \& } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ are } \underline{\text{linearly independent}}.$$

$$\text{So } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ form a basis of } V.$$

$$\text{and } \underline{\dim(V) = 3}.$$

2. Let a linear map  $T : P_2 \rightarrow \mathbb{R}^2$  be defined by  $T(f(t)) = \begin{bmatrix} f(1) \\ f(2) \end{bmatrix}$ .

(1)(8 pts) Find a basis of  $\ker(T)$  and the nullity of  $T$  (that is, the dimension of  $\ker(T)$ ).

$$T \sim A = \begin{bmatrix} [T(1)]_2 & [T(t)]_2 & [T(t^2)]_2 \\ \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\text{So } \ker(A) = \text{Span} \left( \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right)$$

$$\text{So } \ker(T) = \text{Span} (2 - 3t + t^2)$$

$$= \text{Span} (2 - 3t + t^2).$$

&  $2 - 3t + t^2$  is already indep. by itself.

$$\text{So } \boxed{\text{a basis of } \ker(T) = \{2 - 3t + t^2\}}$$

$$\text{and } \boxed{\text{nullity of } T = 1}$$

(2)(3 pts) By using (1), what is the rank of  $T$ ? (that is, what is the dimension of the image of  $T$ ?) You don't have to find a basis of the image of  $T$  but explain how you get your answer.

By rank-nullity theorem, nullity of  $T$  + rank of  $T$  =  $\dim(P_2)$

$$\Rightarrow 1 + \text{rank of } T = 3$$

$$\Rightarrow \boxed{\text{rank of } T = 2}$$



5. (7 pts) Find an orthonormal basis of the subspace  $V$  of  $\mathbb{R}^3$  spanned by two vectors

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

$\vec{v}_1$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{1+4+4}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

$$\begin{aligned} \vec{v}_2^\perp &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{3}(3+2+4) \cdot \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{1}{\sqrt{4+1+0}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

So an orthonormal basis of  $V$  is

$$\vec{u}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \& \quad \vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

6.(7 pts) Let the subspace  $V$  of  $\mathbb{R}^3$  be spanned by two vectors  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ . Find the matrix of the orthogonal projection onto  $V$ . (You may use an orthonormal basis of  $V$  or least-squares solutions  $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$  of the equation  $A\vec{x} = \vec{b}$  by recalling a relation between the orthogonal projection and  $\vec{x}^*$ .)

The given vectors are already orthonormal basis of  $V$ .

$$\begin{aligned} \text{So } \text{Proj}_V &= Q Q^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \text{or } \text{Proj}_V &= A \vec{x}^* = A (A^T A)^{-1} A^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -1 \end{bmatrix} \left( \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(Extra-credit problem, 5 pts, no partial credit)

Consider the following statement:

There exists a linear transformation  $T : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^2$  whose kernel is the space of all  $3 \times 3$  symmetric matrices

If this statement is true, then find such a linear transformation and if this statement is false, then explain why it is false.

False!

Cause :  $\dim(\mathbb{R}^{3 \times 3}) = 9 = \text{rank}(T) + \text{nullity}(T)$

If  $\ker(T) = \{3 \times 3 \text{ symmetric matrices}\}$ ,

then  $\text{nullity}(T) = \dim(\text{symmetric matrices})$   
 $= \dim\left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right\}$   
 $= 3$

So we get equality,

$$9 = \text{rank}(T) + 3$$

So  $\text{rank}(T) = 6$ .

But  $\text{Im}(T)$  is a subspace of  $\mathbb{R}^2$ ,

So it can't have dimension bigger than 2.

So  $\text{rank}(T)$  cannot be 6!