Name : Solution

Student ID # :

Math 2270 - 2
Spring 2006
Instructor: Bo-Hae Im

EXAM 1
Thursday, February 9, 2006

Problem	points	score
1	11	
2	6	
3	8	
4	10	
5	10	
EC	5	
Total	45(+5)	

1.Let $A = \begin{bmatrix} 2 & -6 & 5 \\ 1 & -3 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$.

(1)(4 pts) Compute AB.

$$AB = \begin{bmatrix} 2 & -6 & 5 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 - 6\cdot 1 + 5(-1) & 2\cdot 1 - 6\cdot(-1) + 5\cdot 0 \\ 3\cdot 1 - 3\cdot 1 + 0\cdot(-1) & 1\cdot 1 - 3\cdot(-1) + 0 \end{bmatrix} = \begin{bmatrix} -5 & 8 \\ 0 & 4 \end{bmatrix}$$

(2)(7 pts) Find a basis of $\mathbf{ker}(A)$ and a basis of $\mathbf{im}(A)$.

$$A = \begin{bmatrix} 2 & -6 & 5 \\ 1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \to \left[\begin{array}{ccc:c} \textcircled{1} & \overset{\text{free}}{-3} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{array}\right]$$

If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in ker(A), $\begin{cases} x_1 = 3s \\ x_2 = s \\ x_3 = 0 \end{cases}$ for free s. $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

So ker(A) has a basis $\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right)$. $\Rightarrow \dim(\ker(A)) = 1$.

Also, 1st & 3rd columns have leading 1's.

So im(A) has a basis consisting of 1st & 3rd columns,

$$\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right). \Rightarrow \dim(\operatorname{im}(A)) = 2.$$

2. (6 pts) Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent vectors in $\mathbb{R}^n$.
Prove that $\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_1$ are linearly independent.

Suppose the dependence relation of $\vec{v_1}+\vec{v_2}$, $\vec{v_2}+\vec{v_3}$ & $\vec{v_1}$.

Suppose $c_1(\vec{v_1}+\vec{v_2}) + c_2(\vec{v_2}+\vec{v_3}) + c_3\vec{v_1} = \vec{0}$.

$\Rightarrow (c_1+c_3)\underbrace{\vec{v_1}}_{} + (c_1+c_2)\underbrace{\vec{v_2}}_{} + c_2\cdot\underbrace{\vec{v_3}}_{} = \vec{0}$ ← dependence relation of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

Since $\vec{v_1}, \vec{v_2}$ & $\vec{v_3}$ are linearly independent,

each coefficient must be zero.

$$\Rightarrow \begin{cases} c_1+c_3=0 \\ c_1+c_2=0 \\ c_2=0 \end{cases} \Rightarrow \begin{cases} c_1=0, \\ c_2=0 \\ c_3=0 \end{cases}$$

So the dependence relation insists $c_1=0, c_2=0, c_3=0$.

Therefore they $(\vec{v_1}+\vec{v_2}, \vec{v_2}+\vec{v_3}, \vec{v_1})$ are linearly independent.

3. (8 pts) Let $V = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 | x_1 + 2x_2 - x_3 = 0\}$.

(1) Show that V is a subspace of $\mathbb{R}^3$.
(2) Find a basis of V.
(3) Find $\dim(V)$.

(1) For $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in V, $x_1 = -2x_2 + x_3$. $\Rightarrow$ $[1, -2, 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

So it is in $\ker([1,-2,1])$.

Therefore V is a subspace of $\mathbb{R}^3$.

(2) Since $x_1 = -2\underbrace{x_2 + x_3}_{\text{free}}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $\text{Span}\Big(\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{linearly independent.}}\Big)$.

So a basis of $V = \left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$.

(3) $\dim(V) =$ # of vectors in a basis $= \boxed{2}$.

4. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}-2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}.$$

Let $\mathcal{B} = \left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right)$ be a basis of $\mathbb{R}^2$.

(1)(5 pts) Find the $\mathcal{B}$-matrix of T.

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix} - \begin{bmatrix}2\\-1\end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}0\\-1\end{bmatrix}.$$

$$T\left(\begin{bmatrix}-2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}1\\-1\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

So the $\mathcal{B}$-matrix of $T = \left[(T\begin{bmatrix}1\\-1\end{bmatrix})_{\mathcal{B}} \; (T\begin{bmatrix}-2\\1\end{bmatrix})_{\mathcal{B}} \right] = \boxed{\begin{bmatrix}0 & -1\\-1 & -1\end{bmatrix}}.$

(2)(5 pts) Find $T\left(\begin{bmatrix}8\\-5\end{bmatrix}\right)$.

Let $S = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix}1 & -2\\-1 & 1\end{bmatrix}$

We find the standard matrix $A = SBS^{-1} = \begin{bmatrix}1 & -2\\-1 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\-1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}^{(-1)}$

$= \begin{bmatrix}-3 & -5\\+1 & 2\end{bmatrix}$ //

So $T\left(\begin{bmatrix}8\\-5\end{bmatrix}\right) = A\begin{bmatrix}8\\-5\end{bmatrix} = \begin{bmatrix}-3 & -5\\+1 & 2\end{bmatrix}\begin{bmatrix}8\\-5\end{bmatrix} = \boxed{\begin{bmatrix}1\\-2\end{bmatrix}}$

(Another way is, $\begin{bmatrix}8\\-5\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2\\-3\end{bmatrix}$. So $(T\begin{bmatrix}8\\-5\end{bmatrix})_{\mathcal{B}} = B \cdot \begin{bmatrix}8\\-5\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}0 & -1\\-1 & -1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}$

$= \begin{bmatrix}3\\1\end{bmatrix} \Rightarrow T\begin{bmatrix}8\\-5\end{bmatrix} = 3\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}-2\\1\end{bmatrix} = \boxed{\begin{bmatrix}1\\-2\end{bmatrix}}$)

5. (each 2 pts) Determine whether the following statements are true or false. (No partial credit will be given. You don't have to justify your answer but just write **T** for a true statement and **F** for a false statement.)

(1) If an $n \times n$ matrix A satisfies $A^2 = O_n$, then $A = O_n$. F

$\left(\text{Ex. } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \right)$

(2) If A is a $8 \times \textcircled{5}$ matrix and $rk(A) = 5$, then $ker(A) = \{\vec{0}\}$. T

$\left(\Rightarrow \dim(\ker(A)) = 5-5 = 0. \Rightarrow \ker(A) = \{\vec{0}\}. \right)$

(3) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span the space $\mathbb{R}^m$, then m must be equal to n. F

$(m \leq n.)$

(4) Any square matrix whose column vectors are linearly independent must be invertible. T

$\left(\Rightarrow \ker = \vec{0}, \text{ \& square } \Rightarrow \text{invertible} \right)$

(5) If a subspace V of $\mathbb{R}^n$ contains the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, then $V = \mathbb{R}^n$. T

(Extra-credit problem, 5 pts, NO partial credit. The full credit will be given only if the proof is a mathematically complete one.)

Consider an $m \times n$ matrix A and an $n \times m$ matrix B such that $AB = I_m$. Here assume that $n \neq m$.

Decide which matrix, (A or B or both,) must have linearly independent column vectors, and justify your answer by proving or giving a counterexample.

Answer: Only B has linearly

but A may not have linearly independent column vectors.

<u>Proof</u>: For B, need to show $\ker(B) = \{\vec{0}\}$.

Suppose $B\vec{x} = \vec{0}$. $\Rightarrow \underset{\vec{x}\,=\,I_m\vec{x}\text{ by assumption.}}{\underset{\parallel}{AB\vec{x}}} = A\vec{0} = \vec{0}$.

$\Rightarrow \vec{x} = \vec{0}$. So $\underline{\underline{\ker(B) = \{\vec{0}\}}}$

So columns of B are independent.

For A, need to give an example,

let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $A \cdot B = [1]_{1\times 1}$.

But A has <u>only</u> <u>one column</u> independent!

(Note that if $n = m$, then we have shown in class

that both A & B are invertible,

So both have indep. column vectors.)