Name : Solution

Student ID # :________________________

Honor 2201

Fall 2005

Instructor: Bo-Hae Im

Final Exam

Tuesday, December 13, 2005

Problem	points	score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	10	
8	10	
Total	50	

1. Determine the following infinite series $\sum_{k=1}^{\infty} \frac{2}{k^3}$ is convergent of divergent and justify your answer.

Use the integral test.

$$\int_1^\infty \frac{2}{x^3}\,dx = 2\int_1^\infty x^{-3}\,dx = 2\cdot\left[\frac{x^{-2}}{-2}\right]_1^\infty$$

$$= \left[-\frac{1}{x^2}\right]_1^\infty$$

$$= -\frac{1}{\infty^2} - \left(-\frac{1}{1^2}\right)$$

$$= -\cancel{\frac{1}{\infty}}^{\,0} + \frac{1}{1}$$

$$= 1 \quad \text{(this is a finite number.)}$$

Since $\int_1^\infty \frac{2}{x^2}\,dx$ converges to 1,

by the integral test, $\boxed{\sum_{k=1}^{\infty} \frac{2}{k^3} \text{ converges.}}$

(but it doesnot converge to 1. but to some finite number.)

2. Each morning, a patient receives a 25mg injection of a drug, and 40% of the drug remains in the body after 24 hours. Find the quantity of the drug in the body in the long term right after the injection.

Let B_n be the amount of a drug remaining in the body right after the nth injection.

Then
$$B_1 = 25$$
$$B_2 = 25 + (0.4)\cdot 25$$
$$B_3 = 25 + (0.4)\cdot B_2 = 25 + (0.4)(25 + (0.4)25)$$
$$= 25 + (0.4)\cdot 25 + (0.4)^2\cdot 25$$
$$B_4 = 25 + (0.4)\cdot B_3 = 25 + (0.4)\left(25 + (0.4)25 + (0.4)^2\cdot 25\right)$$
$$= 25 + (0.4)25 + (0.4)^2\cdot 25 + (0.4)^3\cdot 25.$$
$$\vdots$$

This is a geometric series with $B_n = 25 + (0.4)25 + \cdots + (0.4)^{n-1}\cdot 25.$

Since $r = 0.4$ in between 1 & -1, the infinite series

$$\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=1}^{\infty} 25\cdot(0.4)^{k-1} \text{ converges to } \frac{a}{1-r} = \frac{25}{1-0.4}$$

$$= \frac{25}{0.6} = \boxed{\frac{125}{3}}\ \text{mg.} \quad (\approx 41.67\ \text{mg})$$

3. A machine lasts up to 3 years. The figure shows the density function $p(t) = \dfrac{t^2}{9}$ for the length of time it lasts. Find the median time with $p(t)$.

Find T such that $\int_{-\infty}^{T} p(t)\,dt = 0.5$.

Since $p(t) = \frac{t^2}{9}$ for $0 \le t \le 3$ & $p(t) = 0$ otherwise,

$$\int_{-\infty}^{T} p(t)\,dt = \int_{0}^{T} p(t)\,dt.$$

$$\Leftrightarrow \int_0^T \frac{t^2}{9}\,dt = 0.5 \Leftrightarrow \left[\frac{1}{9}\cdot\frac{t^3}{3}\right]_0^T = 0.5$$

$$\Leftrightarrow \frac{T^3}{27} - \frac{0^3}{27} = \frac{1}{2}$$

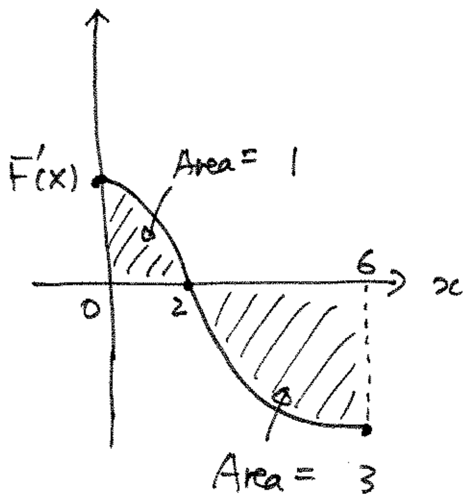
$$\Leftrightarrow \frac{T^3}{27} = \frac{1}{2}$$

$$\Leftrightarrow T^3 = \frac{27}{2}$$

$$\Leftrightarrow T = \sqrt[3]{\frac{27}{2}} = \boxed{\frac{3}{\sqrt[3]{2}}} \approx \boxed{2.38 \text{ years}}$$

years

4. The following is the graph of $F'(x)$ over the interval $[0, 6]$ of the function $F(x)$. If $F(0) = 2$, find the minimum value of $F(x)$ over $[0, 6]$.

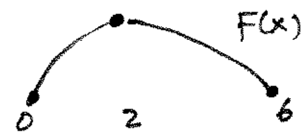


$F'(x) > 0$ for $0 < x < 2$

$F'(x) < 0$ for $2 < x < 6$.

So F is increasing until $x = 2$ and decreasing after that.

A rough graph of F is like



So the minimum occurs one of the end points.

We need to compare $F(0)$ & $F(6)$.

We know $F(0) = 2$. By Fundamental theorem of Calculus,

$$F(6) = F(0) + \int_0^6 F'(x)\,dx$$

$$= 2 + (\ ) - (\ )$$ ∵ Since F' is below the x-axis.

$$= 2 + 1 - 3$$

$$= 0$$

Since $F(0) = 2 > F(6) = 0$, $F(6) = 0$ is the minimum value of F over $[0, 6]$.

5. Solve the differential equation $P'(x) = -P(x)+2e^{-2x}$, when $P(0) = 1$. (You might need the property $e^a e^b = e^{a+b}$ and $\frac{e^a}{e^b} = e^{a-b}$ to simplify your expression while you compute.)

① $e^{-\int f(x)dx} = e^{-\int -1\, dx} = e^{x}$ ($e^{x-2x} = e^{-x}$)

② $e^x \cdot p'(x) = -e^x p(x) + 2 \cdot e^x \cdot e^{-2x}$

$e^x \cdot p'(x) + e^x \cdot p(x) = 2 \cdot e^{-x}$

$(e^x \cdot p(x))' = 2 \cdot e^{-x}$

③ $e^x \cdot p(x) = \int 2 \cdot e^{-x}\, dx + C$

$e^x \cdot p(x) = -2e^{-x} + C$

④ $p(x) = \dfrac{-2e^{-x} + C}{e^x} = -2 \cdot e^{-x} \cdot e^{-x} + C \cdot e^{-x}$

$\Rightarrow p(x) = -2 \cdot e^{-2x} + C \cdot e^{-x}$.

⑤ Since $p(0) = 1$ must be true, $p(0) = -2 \cdot e^0 + C \cdot e^0 = 1$

$\Leftrightarrow -2 \cdot 1 + C \cdot 1 = 1 \Leftrightarrow \boxed{C = 3}$

So $\boxed{p(x) = -2 \cdot e^{-2x} + 3 \cdot e^{-x}}$ is the solution.

6. Let $f(x) = e^{-2x} - 3\sin x$ be a function. Find the tangent line of $y = f(x)$ at $x = 0$.

We need to know the slope of the tangent line at $(0, f(0))$.

Slope $= f'(0) = (-2e^{-2x})\Big|_{x=0} = -2 \cdot e^0 = -2$

Slope $= \boxed{-2}$ & the point $= (0, f(0)) = (0, e^0 - 3)$

$= (0, 1-3) = \boxed{(0, -2)}$

$y - y_1 = m(x - x_1) \Rightarrow y - (-2) = -2(x - 0)$

$\Rightarrow y + 2 = -2x$

$\Leftrightarrow \boxed{y = -2x - 2}$

7. (each 5 pts) Let $f(x) = -x^3 + 3x - 2$ for $-2 \le x \le 2$ be a given function.

(1) Find the critical points and inflection points of f given above if any and justify your answer.

$f'(x) = -3x^2 + 3 = 0 \Leftrightarrow -3(x^2-1)=0 \Leftrightarrow x^2-1=0 \Leftrightarrow \underline{x = \pm 1}$

So the critical points are $\boxed{x=1 \ \& \ x=-1}$.

$f''(x) = -6x = 0 \Rightarrow x=0.$ need to check the concavity.

	-1	0	1
f''	+		−

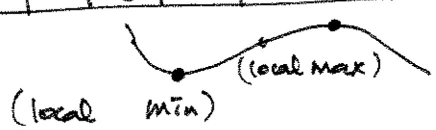
f changes the concavity at $x=0$.

So $\boxed{x=0}$ is an inflection point of f.

(2) Find a local maximum/minimum, and the global maximum/minimum of f given above and justify your answer.

	-2		-1		1		2
f'		−	0	+	0	−	
f		↘		↗		↘	

$f''(x) = -6x$ by (1).



Since $f''(-1) = 6 > 0$, f is CU at $x=-1$ $\Rightarrow$ local min at $(-1, f(-1))$.

Since $f''(1) = -6 < 0$, f is CD at $x=1$. $\Rightarrow$ local max at $(1, f(1))$.

$f(1) = -1+3-2=0$ & $f(-1) = 1-3-2 = -4$

So f has a local min at $\boxed{(-1,-4)}$ & a local max at $\boxed{(1,0)}$.

$f(-2) = 8-6-2 = 0$

$f(-1) = -4$

$f(1) = 0$

$f(2) = -8+6-2 = -4$

$\Rightarrow$ the global max at $\boxed{(-2,0) \text{ or } (1,0)}$

& the global min at $\boxed{(-1,-4) \text{ or } (2,-4)}$.

8. (each 5 pts) Find the following (show all of your work).

(1) $\int x \ln x dx =$

Integration by parts:

$u = \ln x \Rightarrow u' = \frac{1}{x}$

$v' = x \Rightarrow v = \frac{x^2}{2}$

$$\Rightarrow \int x \ln x dx = uv - \int u' v dx$$

$$= \frac{x^2}{2} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2}{2} \cdot \ln x - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2} \cdot \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C$$

$$= \boxed{\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} + C}$$

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(2) $\int 2x \cdot \cos(x^2 + 1) dx =$

Integration by substitution:

$u = x^2 + 1$

$\Rightarrow du = 2x \cdot dx$

$$\Rightarrow \int 2x \cdot \cos(x^2+1) dx = \int \cos(u) \; du$$

$$= \int \cos(u) du$$

$$= \sin u + C$$

$$= \boxed{\sin(x^2+1) + C}$$

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