Math 1100 Test #2

Instructions:

1. Do not look at this test until the strange guy with the red hair says it’s okay to do so.

2. No calculators, books, or other aids are permitted for this test.

3. You will have until **10:30pm** to finish the test. I will remind you when you have 20 min, 5 min, and 2 min remaining.

4. Please remain in your seat if you finish the test in the last 5 minutes so that your peers are not distracted in the final minutes of the test.

5. Please raise your hand if you have any questions and/or would like some more paper.

6. Please show all your work for full credit.

7. Solutions to this test will be posted on-line following class and the test will be returned next class.

8. You are strongly encouraged to celebrate after finishing the test.

<table>
<thead>
<tr>
<th>Question</th>
<th>Max Marks</th>
<th>Earned Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a,b,d</td>
<td>4,5,+5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5 each (20 total), +4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td></td>
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<tr>
<td>6</td>
<td>7</td>
<td></td>
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<tr>
<td>7</td>
<td>+5</td>
<td></td>
</tr>
<tr>
<td>Silly Bonus</td>
<td>+1</td>
<td></td>
</tr>
<tr>
<td>Presentation</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

Bolded numbers are bonus marks.

Total = \frac{60}{60}
1. Differentiate the following functions and simplify your answer.

(a) 

\[ y = x^3 e^{-5x^2} \]

\[ y' = 3x^2 e^{-5x^2} - 10x^4 e^{-5x^2} = x^2 e^{-5x^2} (3 - 10x^2) \]

(b) 

\[ y = \ln\left(\frac{x^2 - 4}{(x + 1)^3}\right) = \ln (x^2 - 4) - \frac{5}{3} \ln (x + 1) \]

\[ y' = \frac{2x}{x^2 - 4} - \frac{5}{3(x + 1)} = \frac{x^2 + 6x + 20}{3(x^2 - 4)(x + 1)} \]

(c) BONUS

\[ y = \log_7(x^2(5 - x)) = \frac{1}{\ln 7} [2 \ln x + \ln(5 - x)] \]

\[ y' = \frac{1}{\ln 7} \left[ \frac{2}{x} + \frac{1}{x - 5} \right] = \frac{1}{\ln 7} \cdot \frac{3x - 10}{x(x - 5)} \]

2. Find the equation of the tangent line to the curve at \( x = 0, y = 3 \)

\[ \frac{1}{x + 1} + 2x^3 y^3 = e^x + \frac{y^2}{9} - 1 \]

We need to find \( \frac{dy}{dx} \), so we differentiate the above equation to get:

\[-(x + 1)^{-2} + 6x^2 y^3 + 6x^3 y^2 \frac{dy}{dx} = e^x + \frac{2}{9} y \frac{dy}{dx} \]

Now we set \( x = 0, y = 3 \) and solve for \( \frac{dy}{dx} \)

\[-1 + 0 + 0 = 1 + \frac{2}{9} \cdot 3 \frac{dy}{dx} \]

\[ \rightarrow \frac{dy}{dx} = -3 \]

Therefore, the equation of the tangent line is:

\[ y = -3x + 3 \]
3. Integrate the following and simplify your answer.

(a) \[
\int \left(x^3 + \frac{8}{x^3} + 5\sqrt[4]{x}\right) \, dx = \frac{1}{4}x^4 - 4x^{-2} + 4x^{\frac{5}{4}} + C
\]

(b) \[
\int \frac{x^2 - 4x}{\sqrt{x^3 - 6x^2} + 2} \, dx = \frac{2}{3}\sqrt{x^3 - 6x^2} + 2 + C
\]

(c) \[
\int \frac{x^2 + 1}{x^3 + 3x + 17} \, dx = \frac{1}{3}\ln(x^3 + 3x + 17) + C
\]

(d) \[
\int_1^3 \frac{x}{e^{3x^2-2}} \, dx = -\frac{1}{6}e^{2-3x^2}\bigg|_1^3
\]
\[
= -\frac{1}{6}(e^{-25} - e^{-1})
\]

(e) **Bonus**

\[
\int 2x \, e^{x^2}(1 + x^2) \, dx = x^2 \, e^{x^2} + C
\]

4. If the monthly marginal cost for a product is \(MC = x^2 + 10\) and the related fixed costs are $50, find the total cost function for the month.

\[C(x) = \frac{1}{3}x^3 + 10x + 50\]

5. A kite is 40ft high and is moving horizontally at a rate of 10 ft/min. If the kite string is taught, at what rate is the string being played out when 50 ft of string is out?

Let \(x\) be the horizontal distance between the kite and the person holding it. We know that

\[
\frac{dx}{dt} = 10 \text{ft/min}
\]

\[
L^2 = x^2 + 40^2
\]

So, when \(L = 50\), we have \(x = \sqrt{50^2 - 40^2} = 30\). To find \(\frac{dL}{dx}\), we differentiate the main equation above to get:

\[
2L \frac{dL}{dx} = 2x \frac{dx}{dt} + 0
\]

\[
\Rightarrow \frac{dL}{dx} = \frac{x \frac{dx}{dt}}{L \frac{dt}{dx}} = \frac{30 \cdot 10}{50} = 6
\]

The string is going out at a rate of 6ft/min.
6. Suppose the demand for a product is given by:

\[ p^2q = 750 - 10q^2 \]

(a) Find the elasticity when \( p = $10, q = 5 \)

\[
2pq + p^2 \frac{dq}{dp} = -20q \frac{dq}{dp}
\]

\[
\rightarrow 100 + 100 \frac{dq}{dp} = -100 \frac{dq}{dp}
\]

\[
\frac{dq}{dp} = -\frac{1}{2}
\]

Therefore,

\[
\eta = -\frac{p \frac{dq}{dp}}{q \frac{dq}{dp}}
\]

\[
= -\frac{10}{5} \left( -\frac{1}{2} \right) = 1
\]

(b) What type of elasticity is this?

Unitary elasticity.

(c) How would revenue be affected by a price increase?

Revenue is unaffected by a change in price.

7. BONUS: Find the area under \( f(x) = 3x^2 - 4x - 4 \) from \( x = 0 \) to \( x = 3 \). The function is plotted below for your convenience.

First note that \( f(x) < 0 \) for \( 0 < x < 2 \) and \( f(x) > 0 \) for \( 2 < x < 3 \). Also note that,

\[
\int f(x) \, dx = x^3 - 2x^2 - 4x + C
\]

Therefore, the area under the curve is:

\[
A = \int_0^3 |f(x)| \, dx
\]

\[
= -\int_0^2 f(x) \, dx + \int_2^3 f(x) \, dx
\]

\[
= -(x^3 - 2x^2 - 4x) \bigg|_0^2 + (x^3 - 2x^2 - 4x) \bigg|_2^3
\]

\[
= 8 + 5
\]

\[
= 13
\]

Silly Bonus: Answer the following for +1 bonus marks.

What country has the highest concentration of Canadians per capita?

(a) Canada

(b) Mexico

(c) USA