Math 1100 Final Exam (EEK!!)

Name: SOLUTIONS

UID: ________________

Instructions:

1. Do not look at this final exam until the strange guy with the red hair says it’s okay to do so.

2. This final exam is worth 40% of your final grade. Solutions will be posted on the course web site by tonight at the latest.

3. No calculators, books, or other aids are permitted for this exam.

4. You will have until 10:00am to finish. I will remind you when you have 45 min, 20 min, and 5 min remaining.

5. Please remain in your seat if you finish the exam in the last 5 minutes so that your peers can finish the exam undistracted.

6. Please raise your hand if you have any questions and/or would like some more paper.

7. Please show all your work for full credit and hand in your scratch paper if necessary.

8. You are strongly encouraged to celebrate after finishing the exam.

<table>
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<tr>
<th>Question</th>
<th>Max Marks</th>
<th>Earned Marks</th>
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<td>Silly Bonus</td>
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<tr>
<td>Presentation</td>
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</tbody>
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Bolded numbers are bonus marks.

Total = \frac{120}{120}
1. Evaluate the following limit:

\[
\lim_{x \to \frac{1}{2}} \frac{4x + 2}{4x^2 + 1} = \frac{4\left(-\frac{1}{2}\right) + 2}{4\left(-\frac{1}{2}\right)^2 + 1} = \frac{-2 + 2}{\frac{1}{4} + 1} = 0
\]

2. Use the definition of the derivative to find \(f'(x)\) where:

\[
f(x) = 3x^2 - 2x
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x + h)^2 - 2(x + h) - 3x^2 + 2x}{h}
\]
\[
= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h}
\]
\[
= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}
\]
\[
= \lim_{h \to 0} 6x + 3h - 2
\]
\[
= 6x - 2
\]

3. Differentiate the following functions and simplify your answer fully.

(a)

\[
f(x) = \frac{2}{3}(5x^3 - 8x^5 - 6)^3
\]
\[
f'(x) = 2(5x^3 - 8x^5 - 6)^2(15x^2 - 40x^4)
\]
\[
= 10x^2(5x^3 - 8x^5 - 6)^2(3 - 8x^2)
\]

(b)

\[
y = \frac{x^3 - 4}{2x^2 + x - 5}
\]
\[
y' = \frac{3x^2(2x^2 + x - 5) - (x^3 - 4)(4x + 1)}{(2x^2 + x - 5)^2}
\]
\[
= \frac{6x^4 + 3x^3 - 15x^2 - 4x^4 - x^3 + 16x + 4}{(2x^2 + x - 5)^2}
\]
\[
= \frac{2x^4 + 2x^3 - 15x^2 + 16x + 4}{(2x^2 + x - 5)^2}
\]

(c)

\[
f(x) = (x + 2)(3x^2 + 1)^\frac{4}{3}
\]
\[
f'(x) = (3x^2 + 1)^\frac{4}{3} + (x + 2) \left(\frac{4}{3}\right)(3x^2 + 1)^\frac{1}{3}(6x)
\]
\[
= (3x^2 + 1)^\frac{1}{3}(3x^2 + 1 + 8x^2 + 16x)
\]
\[
= (3x^2 + 1)^\frac{1}{3}(11x^2 + 16x + 1)
\]
(d) Find \( \frac{dy}{dx} \)

\[
7x^2 + \ln(3x^2y^4) = y - 8
\]

\[
7x^2 + \ln(3) + 2\ln(x) + 4\ln(y) = y - 8
\]

\[
14x + 0 + \frac{2}{x} + \frac{4\,dy}{y\,dx} = \frac{dy}{dx}
\]

\[
x(y - 4)\frac{dy}{dx} = 2y(7x + 1)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{2y(7x + 1)}{x(y - 4)}
\]

(e) Find one of \( f_x \) or \( f_y \). Find the remaining partial derivative for \( \pm 3 \) BONUS.

\[
f(x, y) = 4x^3 - 5\frac{y^5}{x} + \frac{1}{3}y^2
\]

\[
f_x = 12x^2 + \frac{5y^5}{x^2}
\]

\[
f_y = -25\frac{y^4}{x} + \frac{2}{3}y
\]

(f) \( \pm 4 \) BONUS

\[
y = \frac{e^{3x}}{\sqrt{x^2 - 2}}
\]

\[
y' = \frac{1}{x^2 - 2} \left( 3 e^{3x} \sqrt{x^2 - 2} - e^{3x} \left( \frac{1}{2} \right) (x^2 - 2)^{-\frac{1}{2}} (2x) \right)
\]

\[
= \frac{e^{3x}}{(x^2 - 2)^{\frac{3}{2}}} \left( 3(x^2 - 2) - x \right)
\]

\[
= \frac{e^{3x}(3x^2 - x - 6)}{(x^2 - 2)^{\frac{3}{2}}}
\]

4. Integrate the following and simplify your answer.

(a)

\[
\int_0^4 3x^2\sqrt{4 + x^3} \, dx = \frac{2}{3} (4 + x^3)^{\frac{3}{2}} \bigg|_0^4
\]

\[
= \frac{2}{3} \left[ 64^{\frac{3}{2}} - 4^{\frac{3}{2}} \right]
\]

(b)

\[
\int \frac{4x^4 - 12x^2 - 5x}{x^2 - 3} \, dx = \int \frac{4x^2(x^3 - 3) - 5x}{x^2 - 3} \, dx
\]

\[
= \int \left( 4x^2 - \frac{5x}{x^2 - 3} \right) \, dx
\]

\[
= \frac{4}{3}x^3 - \frac{5}{2}\ln(x^2 - 3) + C
\]
(c) \[
\int_{-1}^{1} 5x(3x^2 - 4)^5 \, dx = \frac{5}{6} \left[ \frac{1}{6}(3x^2 - 4)^6 \right]_{-1}^{1} \\
= \frac{5}{36} \left[ (3 - 4)^6 - (3 - 4)^6 \right] \\
= 0
\]

(d) \[
\int_{-\infty}^{\infty} 3x^3 e^{-x^4} \, dx = \lim_{a \to -\infty} \int_{a}^{c} 3x^3 e^{-x^4} \, dx + \lim_{b \to \infty} \int_{c}^{b} 3x^3 e^{-x^4} \, dx \\
= \lim_{a \to -\infty} \left[ -\frac{3}{4} e^{-x^4} \right]_{a}^{c} + \lim_{b \to \infty} \left[ -\frac{3}{4} e^{-x^4} \right]_{c}^{b} \\
= \lim_{a \to -\infty} \left[ -\frac{3}{4} e^{-a^4} \right] - \lim_{b \to \infty} \left[ -\frac{3}{4} e^{-b^4} \right] \\
= 0 + 0 = 0
\]

(e) (+4 BONUS) \[
\int \frac{7 e^{4\sqrt{x}}}{\sqrt{x}} \, dx = \frac{7}{2} e^{4\sqrt{x}} + C
\]

5. T-mobile is selling annoying cell phone jingles over the holiday season and finds that the monthly costs of production are \( C(x) = \frac{1}{3}x^3 - 7x^2 + 24x + 150 \), where \( x \) is the number of (thousand) jingles. How many jingles should T-mobile sell to minimize its costs if the company can sell at most 10,000 jingles due to customer complaints?

Note that \( C'(x) = x^2 - 14x + 24 = (x - 12)(x - 2) \)

Therefore, \( C' = 0 \) when \( x = 12, 2 \). Because \( x = 12 \) is out of the domain for production, we need to evaluate the cost at \( x = 0, 2, 10 \).

\[
C(0) = 150 \\
C(2) = \frac{8}{3} - 28 + 48 + 150 = 172.66 \\
C(10) = \frac{1000}{3} - 700 + 240 + 150 = 23.33
\]

So the costs are minimized at \( x = 10 \).
6. If the demand and supply functions for a product are \( p = 70 - q^2 \) and \( p = 10 + \frac{3}{2}q \), respectively, find the tax per unit \( t \) that will maximize the tax revenue \( T \).

First we want to see what the equilibrium quantity will be after the tax has been applied. To this end, let:

\[
70 - q^2 = 10 + \frac{3}{2}q + t
\]

\[\rightarrow t = 60 - \frac{3}{2}q - q^2\]

Define: \( T = qt = 60q - \frac{3}{2}q^2 - q^3 \)

Set: \( \frac{dT}{dq} = 0 \Rightarrow 60 - 3q - 3q^3 = -3(q + 5)(q - 4) \)

\[\rightarrow q = 4\]

\[\Rightarrow t = 60 - \frac{3}{2} \cdot 4 - 4^2 = 38\]

7. A plot of the marginal revenue vs. number of units sold is shown in the figure below. Use this graph to answer the following questions:

(a) What can you say about the revenue between points A and B?
   The revenue is increasing (at a decreasing rate).

(b) What point(s) indicated on the graph is the revenue maximized locally (if at all)?
   B, G

(c) What point(s) indicated on the graph is the revenue minimized locally (if at all)?
   E

(d) What point(s) indicated on the graph is a point of diminishing returns (if at all)?
   F (Note returns are starting to increase at point C and the point at which they start to diminish is at point F)
8. A firm’s marginal cost for a product is \( MC = 5x + 16 \), marginal revenue is \( MR = 40 - 7x \), and total cost of production of 10 items is $500.

(a) Find the optimal level of production.

Note that \( P' = 0 \) when \( MR = MC \), so this means that
\[
5x + 16 = 40 - 7x \quad \rightarrow \quad x = 2
\]

(b) Find the profit function.

We integrate the revenue and cost function separately to find the profit function.

\[
R = 40x - \frac{7}{2}x^2
\]
\[
C = \frac{5}{2}x^2 + 16x + K
\]

To find \( K \), we set \( C(10) = 500 \) and solve for \( K \) to get:
\[
C(10) = 500 = \frac{500}{2} + 160 + K \quad \rightarrow \quad K = 90
\]

Therefore, the profit function is:
\[
P = R - C = 24x - 6x^2 - 90
\]

(c) Find the profit or loss at the optimal level of production.

\[
P(2) = -66
\]

(d) Should production be continued in the short run? Explain.

Yes, because the cost of stopping production is greater than the cost of continuing business, i.e. \(-90 = P(0) < P(2) = -66\).

(e) Should production be continued in the long run? Explain.

No, in the long run the business will lose too much money assuming the market conditions remain the same.

9. Choose either part a or part b to answer. Answer the remaining question for +5 BONUS

(a) Assume that water is being purified by causing it to flow through a conical filter whose height is always three times greater than the radius. If the depth of the water is decreasing at a rate of 2 inches per minute when the depth is 3 inches, at what rate is the volume of water flowing out of the filter at this instant? (Note: \( V = \frac{1}{3}\pi r^2h \))

We are given that \( h = 3r \) and \( \frac{dh}{dt} = -2 \) when \( h = 3 \). We implicitly differentiate the volume equation to get:
\[
V = \frac{\pi}{3^3}h^3
\]
\[
\Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{3^3}3h^2 \frac{dh}{dt}
\]
\[
\Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{3^3}3 \cdot 3^2 \cdot (-2)
\]
\[
= -2\pi
\]

Therefore, the volume is decreasing at a rate of 2\( \pi \) cubic inches per minute.
(b) The supply demand function is \( p = x^2 + 2x \) and the consumer demand function is \( p = \frac{60}{x+1} \). What is the consumer surplus at the equilibrium quantity of 3 units?

The price at equilibrium is \( p = \frac{60}{3+1} = 15 \), therefore, the consumer surplus is:

\[
CS = \int_{0}^{3} \frac{60}{x+1} \, dx - 15 \cdot 3
\]

\[
= 60 \ln(x+1) \bigg|_{0}^{3} - 45
\]

\[
= 60 \ln(4) - 45
\]

10. (\(+5 \text{ BONUS}\)) Suppose that the marginal propensity to consume is

\[
\frac{dC}{dy} = 5 + \frac{\ln(y+1)}{y+1}
\]

where \( C \) is in billions of dollars and where the consumption is $8.65 billion when disposable income is 0. Find the national consumption function.

\[
C(y) = 5y + \frac{\ln(y+1)^2}{2} + K
\]

Set \( C(0) = 8.65 = K \)

\[
\rightarrow C(y) = 5y + \frac{\ln(y+1)^2}{2} + 8.65
\]

11. Find the area between \( f(x) = x^2 + x \) and \( g(x) = x^3 - x \). The functions are plotted below for your reference.

From the plot we see that the functions cross at \( x = -1, 0, 2 \).

Let \( d = f - g = -x^3 + x^2 + 2x \)

where \( d \) is the distance between the two functions. Note that \( d > 0 \) for \( 0 < x < 2 \) and \( d < 0 \) for \(-1 < x < 0 \). Therefore, the area between the two regions is:

\[
A = -\int_{-1}^{0} (-x^3 + x^2 + 2x) \, dx + \int_{0}^{2} (-x^3 + x^2 + 2x) \, dx
\]

\[
= -\left( -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 \right) \bigg|_{-1}^{0} + \left( -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 \right) \bigg|_{0}^{2}
\]

\[
= -\frac{1}{4} + \frac{1}{3} + 1 - \frac{2^4}{4} + \frac{2^3}{3} + 2^2
\]

\[
= \frac{37}{12}
\]
12. Provide a complete sketch of the following function using the steps discussed in class. Note that the first and second derivatives of the function are already provided for you.

\[ f(x) = \frac{10(25 - 6x)}{3(x - 5)^2} \quad f'(x) = \frac{20 \, 3x - 10}{3 \, (x - 5)^3} \quad f''(x) = \frac{20(5 - 2x)}{(x - 5)^4} \]

Step 1: General Observations
Note that \( f \left( \frac{25}{6} \right) = 0 \) and the domain is all \( x \neq 5 \).

Step 2: HA & VA

\[ \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{-60x}{3x^2} = \lim_{x \to \pm \infty} -\frac{20}{x} = 0 \]

There is a VA at \( x = 5 \). Note that \( f(x) < 0 \) for all \( x > \frac{25}{6} \approx 4.167 \), therefore,

\[ \lim_{x \to 5^-} f(x) = -\infty = \lim_{x \to 5^+} f(x) \]

(because the denominator is always positive and the numerator is negative sufficiently close to \( x = 5 \))

Step 3: CP & Max/Min

\( f'(x) = 0 \) when \( x = \frac{10}{3} \approx 3.33 \), and \( f'(x) \) is undefined at \( x = 5 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x &lt; 3.33 )</th>
<th>( x = 3.33 )</th>
<th>( 3.33 &lt; x &lt; 5 )</th>
<th>( x = 5 )</th>
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<td>sign of ( f''(x) )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>DNE</td>
<td>+</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>↑</td>
<td>6</td>
<td>↓</td>
<td>DNE</td>
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Therefore, there is a max at \( x = 3.33 \).

Step 4: POI & Curvature

\( f''(x) = 0 \) when \( x = \frac{5}{2} = 2.5 \), and \( f''(x) \) is undefined at \( x = 5 \). Therefore, we find that:

<table>
<thead>
<tr>
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<th>( x &lt; 2.5 )</th>
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<th>( 2.5 &lt; x &lt; 5 )</th>
<th>( x = 5 )</th>
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<td>0</td>
<td>-</td>
<td>DNE</td>
<td>-</td>
</tr>
<tr>
<td>( y )</td>
<td>( \text{CU} )</td>
<td>( \frac{10}{3} )</td>
<td>( CD )</td>
<td>DNE</td>
<td>( CD )</td>
</tr>
</tbody>
</table>

Step 5: Sketch
Silly Bonus: Circle the correct answer for +2 bonus marks.
How tall is Geoff?

a) 6’ 3”

b) 6 feet and 3 inches

c) 190.5 cm