Solution Manual

5.2

#1 $\cos\theta = \dfrac{z+z^{-1}}{2}$ $\quad z = e^{i\theta} \quad dz = i e^{i\theta} d\theta$

$$\int_\gamma \frac{1}{5 - 4\cdot\frac{z+z^{-1}}{2}} \cdot \frac{1}{iz}\, dz = i \int \frac{1}{2z^2-5z+2}\, dz$$

$$= i \cdot 2\pi i \cdot \operatorname{Res}(f, \tfrac{1}{2}) = -2\pi \cdot \frac{1}{2(\frac{1}{2}-2)} = \frac{2}{3}\pi$$

#2 $\sin\theta = (z - z^{-1})/2i$

$$\int_\gamma \frac{1}{10+6(z-z^{-1})/2i} \cdot \frac{1}{iz}\, dz = \int_\gamma \frac{1}{3z^2+10iz-3}\, dz$$

$$= 2\pi i \operatorname{Res}(f, \tfrac{i}{3}) = 2\pi i \cdot \frac{1}{3(\frac{i}{3}-3i)} = -\frac{1}{4}\pi$$

#3

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\, dx = 2\pi i \operatorname{Res}\left(\frac{1}{z^2+2z+2}, -1+i\right)$$

$$= 2\pi i \frac{1}{(-1+i)-(-1-i)} = \pi$$

#4

$$2\int_0^\infty \frac{x^2}{(1+x^2)^2}\, dx = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\, dx = 2\pi i \operatorname{Res}(f, i)$$

$$\int_0^\infty \frac{x^2}{(1+x^2)^2}\, dx = \pi i \cdot -\frac{i}{4} = \frac{\pi}{4}$$

#5 $\displaystyle\int_{-\infty}^{\infty}$ odd fcn $= 0$

\# 6 $$2\int_0^\infty \frac{1}{1+x^6}\,dx = 2\pi i \sum_{\substack{z=i \\ e^{i\frac{\pi}{6}}}} \operatorname{Res}(f, z)$$

$$\int_0^\infty \frac{1}{1+x^6}\,dx = \pi i \left[\frac{1}{2i\cdot(1+1+1)} + \frac{1}{6(e^{i\frac{\pi}{6}})^5} + \frac{1}{6(e^{i\frac{5\pi}{6}})^5}\right]$$

$$(1+z^6) = (z^2+1)(z^4-z^2+1) = 1+[(z-z_0)+z_0]^6$$

$$= (z_0^6+1) + 6z_0^5(z-z_0) + h.o.t$$

$$= \pi i\left[-\frac{i}{6} + -\frac{e^{i\frac{\pi}{6}}}{6} + -\frac{e^{i\frac{5\pi}{6}}}{6}\right] = \pi i\cdot\frac{1}{6}(-2$$

$$= \frac{\pi}{3}$$

\# 7 $$\int_{t=0}^\infty \frac{1}{1+z^3}\,dz \qquad z = e^{i\frac{2\pi}{3}}t$$

$$dz = e^{i\frac{2\pi}{3}}\,dt$$

$$= e^{i\frac{2\pi}{3}}\int_{t=0}^\infty \frac{1}{1+t^3}\,dt$$

i.e. $$\left(1 - e^{i\frac{2\pi}{3}}\right)\int_{t=0}^\infty \frac{1}{1+t^3}\,dt = 2\pi i \operatorname{Re}\left(\frac{1}{1+z^3}, z_0\right)$$

$$z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$$

$$\int_{t=0}^\infty \frac{1}{1+t^3}\,dt = \frac{1}{3z_0^2}\cdot\frac{1}{1-z_0^2}\cdot 2\pi i = \frac{2}{3}\pi i\cdot\frac{1}{2i\frac{\sqrt{3}}{2}}$$

$$= \frac{2\pi}{3\sqrt{3}}$$

#8 $\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$

$$\left| \frac{e^{i(x+iy)}}{1+z^2} \right| \leq \frac{e^{-y}}{R^2-1} < \frac{1}{R^2-1} \quad \text{since } y>0$$

$$= 2\pi i \ \mathrm{Res}\left(\frac{e^{iz}}{1+z^2},\ i\right)$$

$$= 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

#14 $\int_0^{\infty} \frac{1}{t+t(\log t)^2}\,dt = \int_{x=-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\pi i \cdot \frac{1}{2i} = \pi$

$x = \log t$

§.3

#2 $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{1+x^2} e^{-itx}\,dx$

$$= \sqrt{2\pi}\, i \ \mathrm{Res}\left(\frac{z}{1+z^2} e^{-itz},\ i\right) \quad \text{if } t<0$$

$$= \sqrt{2\pi}\, i \cdot \frac{i}{2i} \cdot e^{t} = \sqrt{\frac{\pi}{2}}\, e^{t}\, i$$

$$\hat{f}(t) = -\sqrt{2\pi}\, i \ \mathrm{Res}\left(\frac{z}{1+z^2} e^{-itz},\ -i\right) \quad \text{if } t>0$$

$$= -\sqrt{2\pi}\, i \cdot \frac{(-i)}{(-2i)}\, e^{-t} = -\sqrt{\frac{\pi}{2}}\, e^{-t}\, i$$

(3#. $=$ (4))

\# 3

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+x^4} e^{-itx}\, dx$$

$$\hat{f}(t) = \sqrt{2\pi}\, i \sum_{z_0} \mathrm{Res}\left(\frac{1}{1+z^4} e^{-itz}, z_0 \right) \quad \text{when } t < 0$$

$$\left[= e^{i\frac{\pi}{4}},\ e^{+i\frac{\pi}{4}3} \right]$$

$$= \sqrt{2\pi}\, i \sum \frac{1}{4 z_0^3} e^{-itz_0}$$

$$\hat{f}(t) = -\sqrt{2\pi}\, i \sum \frac{1}{4 z_0^3} e^{-iz_0 t} \quad \text{when } t > 0$$

$$z_0 = e^{-i\frac{\pi}{4}},\ e^{-i\frac{\pi}{4}3}$$

\# 4

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\, e^{-|x|} e^{-itx}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{x-itx}\, dx + \frac{1}{2} \int_{0}^{\infty} e^{-x-itx}\, dx$$

$$= \frac{1}{2} \frac{1}{1-it} e^{(1-it)x} \Big|_{-\infty}^{0} + \frac{1}{2} \frac{1}{-(1+it)} e^{-(1+it)x} \Big|$$

$$= \frac{1}{2} \cdot \frac{1}{1-it} + \frac{1}{2} \cdot \frac{1}{1+it} = \frac{1}{1+t^2}$$

5

$$\hat{f}(t) = \sqrt{2\pi}\, i \ \operatorname{Res}\left(\frac{1}{z^2+4z+5} \cdot e^{-itz}, -2+i \right) \quad \text{when } t < 0$$

$$= \sqrt{2\pi}\, i \ \frac{1}{2i}\, e^{(1+2i)t}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{(1+2i)t}$$

$$\hat{f}(t) = -\sqrt{2\pi}\, i \cdot \frac{1}{-2i}\, e^{(-1+2i)t} \quad \text{when } t > 0$$

$$= \sqrt{\frac{\pi}{2}}\, e^{(-1+2i)t}$$

7

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\, dx = \int_{-\infty}^{\infty} \frac{x \left(\frac{e^{ix} - e^{-ix}}{2i} \right)}{1+x^2}\, dx$$

$$= \frac{1}{2i} \left[\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\, dx - \int_{-\infty}^{\infty} \frac{x e^{-ix}}{1+x^2}\, dx \right]$$

$$= \frac{1}{2i} \operatorname{Res}\left(\frac{z e^{iz}}{1+z^2}, i \right) \cdot 2\pi i + \frac{1}{2i} \cdot 2\pi i \operatorname{Res}\left(\frac{z e^{-iz}}{1+z^2}, -i \right)$$

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if z is on the upper half plane

$$\frac{|z e^{iz}|}{|1+z^2|} \leq \frac{R e^{-y}}{R^2-1} \qquad z = x+iy$$

Similarly for the lower half

#7 $= \pi \cdot \frac{i}{2i} e^{-1} + \pi \cdot \frac{-i}{-2i} \cdot e^{-1} = \frac{\pi}{e}$ (6)

#10 $\int_0^\infty \frac{x^{\frac{2}{3}}}{1+x}\, dx$ uses Mellin Transform, Thm 5.3.7

or #5.3.8

5.4

#2

$$f(z) = \frac{1}{1+z^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}\left(\sum_{-\infty}^{\infty} \frac{1}{1+n^2} - 1\right) = \frac{e^{-\pi}}{e^{\pi} - e^{-\pi}} = \frac{1}{e^{2\pi} - 1}$$

$$-\sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \sum_{z_0 = \pm i} \mathrm{Res}\left(\frac{1}{1+z^2} \frac{\pi \cos \pi z}{\sin \pi z}, z_0\right)$$

$$= \frac{1}{2i} \frac{e^{-\pi} + e^{\pi}}{e^{-\pi} - e^{\pi}} \cdot i - \frac{1}{2i} \frac{e^{-\pi} + e^{\pi}}{e^{\pi} - e^{-\pi}} \cdot i$$

$$\sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}$$

#4

$$f(z) = \frac{1}{z^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = -\mathrm{Res}\left(\frac{1}{z^4} \frac{\pi \cos \pi z}{\sin \pi z}, 0\right) = \frac{\pi\left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} + \cdots\right)}{\pi z^5\left(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120} \cdots\right)}$$

$$= -\mathrm{Res}\left(\frac{1}{z^5}\left(1 - \frac{\pi^2}{3} z^2 - \frac{\pi^4}{45} z^4\right)\right) = \frac{\pi^4}{45}$$