INSTRUCTOR: H.-PING HUANG

LAST NAME _______________________
FIRST NAME _______________________
ID NO. ____________________________

INSTRUCTION: SHOW ALL OF YOUR WORK. MAKE SURE YOUR ANSWERS ARE CLEAR AND LEGIBLE. USE SPECIFIED METHOD TO SOLVE THE QUESTION. IT IS NOT NECESSARY TO SIMPLIFY YOUR FINAL ANSWERS.

PROBLEM 1 25 __________
PROBLEM 2 25 __________
PROBLEM 3 25 __________
PROBLEM 4 25 __________
TOTAL 100 __________

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PROBLEM 1

(25 pt) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{y^2 - x^2 y}{|y - x^2|} & \text{if } y \neq x^2 \\ 0 & \text{if } y = x^2 \end{cases}$$

At which points of $\mathbb{R}^2$ is this function continuous?
PROBLEM 2

(25 pt) Is the image of a closed set under a continuous function necessarily closed? Prove that it is or give an example where it is not.

Is the image of an open set under a continuous function necessarily open? Prove that it is or give an example where it is not.
PROBLEM 3

(25 pt) Prove that the following limit exists and evaluate it.

\[ \lim_{n \to \infty} \int_{0}^{\pi/2} \sqrt{\frac{x}{n} + \cos \frac{x}{n}} \, dx \]
PROBLEM 4

(25 pt) Des the series \( \sum_{k=0}^{\infty} x^k y^k \) converge uniformly on the square
\[ \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\} \]
Justify your answer.

Des the series \( \sum_{k=0}^{\infty} x^k y^k \) converge uniformly on the disc
\[ \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \]
Justify your answer.
(25 pt) Suppose that $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ for some $n, m \in \mathbb{N}$.

(a) If $T(1, 1) = (3, \pi, 0)$ and $T(0, 2) = (4, 0, 1)$, find the matrix representative of $T$.

(b) If $T(1, 1, 0) = (e, \pi)$, $T(0, -1, 1) = (1, 0)$, and $T(1, 1, -1) = (4, 7)$, find the matrix representative of $T$. 