20. Find the volume of the ellipsoid given by $\frac{x^2}{64}+\frac{y^2}{25}+\frac{z^2}{49}=1$.

$u=\frac{x}{8} \quad v=\frac{y}{5} \quad w=\frac{z}{7}$

$x=8u \quad y=5v \quad z=7w$

The Jacobian is $(8)(5)(7)$

$\int_D 1\,dx\,dy\,dz,$ where D is $\frac{x^2}{64}+\frac{y^2}{25}+\frac{z^2}{49}\le 1$

$(8)(5)(7)\iiint_S 1\,du\,dv\,dw,$ where S is $u^2+v^2+w^2\le 1$

The volume is

$\frac{4}{3}\pi r^3(8)(5)(7)$

The radius is one therefore the volume is

$\boxed{\frac{4}{3}\pi(8)(5)(7)}$

21. Find the area of the ellipse given by $\frac{x^2}{4}+\frac{y^2}{64}=1$.

Subtitute

$$u=\frac{x}{2} \Rightarrow x=2u \Rightarrow dx=2du$$

$$v=\frac{y}{8} \Rightarrow y=8v \Rightarrow dy=8dv$$

Set up the Integral

$\iint_D 1\,dx\,dy$ where D is $\frac{x^2}{4}+\frac{y^2}{64}\le 1$

$\iint_S 1\,(2du)(8dv)$ where S is $u^2+v^2\le 1$

$$16\iint_S 1\,du\,dv$$

$$(16)(\pi)$$

The area of the ellipse is 16π.

22. Consider the transformation T : $x = \frac{14}{50}u - \frac{48}{50}v,\ y = \frac{48}{50}u + \frac{14}{50}v.$

A. Compute the Jacobian.

$$\begin{vmatrix} \frac{14}{50} & \frac{-48}{50} \\ \frac{48}{50} & \frac{14}{50} \end{vmatrix} = \left(\frac{14}{50}\right)\left(\frac{14}{50}\right) - \left(\frac{48}{50}\right)\left(\frac{-48}{50}\right) = \frac{14^2}{50^2} + \frac{48^2}{50^2} = \frac{196}{2500} + \frac{2304}{2500} = 1$$

B. The transformation is linear, which implies that it transforms lines into lines. Thus, it transforms the square $S:\ -50 \le u \le 50,\ -50 \le v \le 50$ into a square $T(S)$ with vertices:

$$T(50,50) = \left(\left(\frac{14}{50}\right)(50) - \left(\frac{48}{50}\right)(50),\ \left(\frac{48}{50}\right)(50) + \left(\frac{14}{50}\right)(50)\right)$$
$$= \left((14-48),\ (48+14)\right) = \boxed{(-34,\ 62)}$$

$$T(-50,50) = \left(\left(\frac{14}{50}\right)(-50) - \left(\frac{48}{50}\right)(50),\ \left(\frac{48}{50}\right)(-50) + \left(\frac{14}{50}\right)(50)\right)$$
$$= \left((-14-48),\ (-48+14)\right) = \boxed{(-62,\ -34)}$$

$$T(-50,-50) = \left(\left(\frac{14}{50}\right)(-50) - \left(\frac{48}{50}\right)(-50),\ \left(\frac{48}{50}\right)(-50) + \left(\frac{14}{50}\right)(-50)\right)$$
$$= \left((-14+48),\ (-48-14)\right) = \boxed{(34,\ -62)}$$

$$T(50,-50) = \left(\left(\frac{14}{50}\right)(50) - \left(\frac{48}{50}\right)(-50),\ \left(\frac{48}{50}\right)(50) + \left(\frac{14}{50}\right)(-50)\right)$$
$$= \left((14+48),\ (48-14)\right) = \boxed{(62,\ 34)}$$

C. Use the transformation T to evaluate the integral $\iint_{T(S)} x^2 + y^2\, dA.$

$$\iint_{T(S)} x^2 + y^2\, dA \ \Rightarrow\ \int_{u=-50}^{50}\int_{v=-50}^{50} \left(\tfrac{14}{50}u - \tfrac{48}{50}v\right)^2 + \left(\tfrac{48}{50}u + \tfrac{14}{50}v\right)^2 |J|\, du\, dv$$

$$\int_{u=-50}^{50}\int_{v=-50}^{50} \left[\left(\tfrac{14}{50}\right)^2 + \left(\tfrac{48}{50}\right)^2\right]u^2 + \left[\left(\tfrac{48}{50}\right)^2 + \left(\tfrac{14}{50}\right)^2\right]v^2 - 2\left(\tfrac{(14)(48)}{50^2}\right)uv + 2\left(\tfrac{(14)(48)}{50^2}\right)uv\ dudv$$

$$\int_{u=-50}^{50}\int_{v=-50}^{50} u^2 + v^2\, du\, dv \ \Rightarrow\ \boxed{(100)\left(\tfrac{1}{3}u^3\Big|_{-50}^{50}\right) + (100)\left(\tfrac{1}{3}v^3\Big|_{-50}^{50}\right)}$$

23. Suppose that $\iint_D f(x,y)\,dA = 2$ where D is the disk $x^2 + y^2 \le 1$. Now suppose E is the disk $x^2 + y^2 \le 4$ and $g(x,y) = 3f\left(\frac{x}{2},\frac{y}{2}\right)$. What is the value of $\iint_E g(x,y)\,dA$?

$$\iint_E 3f\left(\tfrac{x}{2},\tfrac{y}{2}\right)dx\,dy$$

$u = \frac{x}{2} \;\Rightarrow\; x = 2u$ The Jacobian is 4.

$v = \frac{y}{2} \;\Rightarrow\; y = 2v$

$\iint_S 3f(u,v)(2du)(2dv)$ where S is $u^2 + v^2 \le 1$

$$12\iint_S f(u,v)\,du\,dv$$

$$12\iint_D f(x,y)\,dA = (12)(2) = \boxed{24}$$

24. Find the area of the region in the first quadrant bounded by the curves $y^2 = 5x,\ y^2 = 6x,\ x^2 = 5y,\ x^2 = 6y.$

$u = \dfrac{y^2}{x}$ from 5 to 6

$v = \dfrac{x^2}{y}$ from 5 to 6

$u^2 v = y^3 \qquad y = u^{\frac{2}{3}} v^{\frac{1}{3}}$

$uv^2 = x^3 \ \Rightarrow x = u^{\frac{1}{3}} v^{\frac{2}{3}}$

$$J = \begin{vmatrix} \frac{1}{3} u^{\frac{-2}{3}} v^{\frac{2}{3}} & \frac{2}{3} u^{\frac{1}{3}} v^{\frac{-1}{3}} \\ \frac{2}{3} u^{\frac{-1}{3}} v^{\frac{1}{3}} & \frac{1}{3} u^{\frac{2}{3}} v^{\frac{-2}{3}} \end{vmatrix} = \left(\frac{1}{3}\right)^2 - \left(\frac{2}{3}\right)^2 = \frac{-1}{3}$$

$$\int_{u=5}^{6} \int_{v=5}^{6} \frac{1}{3}\, du\, dv = \boxed{\frac{1}{3}}$$

25. Evaluate $\iint_R \left(x^2+y^2\right)dA$ where R is the region in the first quadrant bounded by the curves: $xy=1,\ xy=2,\ y=x$ and $y=3x$.

$u=xy$ from 1 to 2

$v=\dfrac{y}{x}$ from 1 to 3

$uv=y^2 \;\Rightarrow\; y=u^{\frac{1}{2}}v^{\frac{1}{2}}$

$uv^{-1}=x^2 \;\Rightarrow\; x=u^{\frac{1}{2}}v^{\frac{-1}{2}}$

$$\text{J}=\begin{vmatrix} \frac{1}{2}u^{\frac{-1}{2}}v^{\frac{-1}{2}} & \frac{-1}{2}u^{\frac{1}{2}}v^{\frac{-3}{2}} \\ \frac{1}{2}u^{\frac{-1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{\frac{-1}{2}} \end{vmatrix}=\frac{1}{4}v^{-1}+\frac{1}{4}v^{-1}=\frac{1}{2v}$$

$$\int_{u=1}^{2}\int_{v=1}^{3}\left(uv+\frac{u}{v}\right)\frac{1}{2v}\,dv\,du$$

$$\int_{u=1}^{2}\int_{v=1}^{3}\frac{1}{2}u+\frac{1}{2}uv^{-2}\,dv\,du$$

$$\int_{u=1}^{2}\frac{u}{2}v-\frac{u}{2v}\Big|_1^3\,du$$

$$\int_{u=1}^{2}\frac{4u}{3}=\frac{2u^2}{3}\Big|_1^2$$

$\boxed{2}$