HIDDEN STRUCTURE AND COMPUTATION - 2021 PRE-REU FOURIER SERIES AND THE BASEL PROBLEM (2021-06-24)

1. The Basel problem

Exercise 1.0.1. Show that $\sum_{n=1}^{\infty} 1/n^2$ converges.

But what number does it converge to? Pietro Mengoli asked this in 1650, but it took until 1734 before it was solved by Leonhard Euler:

$$\sum_{n=1}^{\infty} 1/n^2 = (\text{... drumroll ...})\ \pi^2/6$$

Today we'll explain one way to make this computation using *Fourier series.*

1.1. **Fourier series.**

If V is a Euclidean space with inner product $\langle , \rangle$, recall that a set of vectors $\vec{v}_1, \dots \vec{v}_n$ are orthonormal if

$$\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Example 1.1.1. The standard basis vectors $\vec{e}_1, \dots \vec{e}_n$ are orthonormal vectors in $\mathbb{R}^n$ with the standard Euclidean inner product $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$.

Exercise 1.1.2. If V is a Euclidean space with inner product $\langle , \rangle$ and $\vec{f}_1, \dots, \vec{f}_n$ are an orthonormal basis, show that for any $\vec{v} \in V$,

$$[\vec{v}]_{\vec{f}_\bullet} = \begin{bmatrix} \langle \vec{v}, \vec{f}_1 \rangle \\ \langle \vec{v}, \vec{f}_2 \rangle \\ \dots \\ \langle \vec{v}, \vec{f}_n \rangle \end{bmatrix}$$

We now consider the Euclidean space $C^0([-\pi, \pi])$ of continuous real valued functions on the closed interval $[-\pi, \pi]$ equipped with the inner product

$$\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx$$

Exercise 1.1.3. Use CoCalc to convince yourself that the functions

$$\frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots$$

are orthonormal for this inner product. You will verify this rigorously in Exercise 1.1.8 if you get there, but for now take it as a fact.

There are some subtleties in making this statement precise for an infinite dimensional vector space, but these are essentially an orthonormal basis.

In particular, we have

Fact 1.1.4. For any $f(x) \in C^0([-\pi, \pi])$, we have

$$f(x) = \langle f(x), \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} + \sum_{n>0} \langle f(x), \cos(nx) \rangle \cos(nx) + \sum_{n>0} \langle f(x), \sin(nx) \rangle \sin(nx)$$

This is the *Fourier series expansion* for $f(x)$.

Exercise 1.1.5. What does this fact have to do with Exercise 1.1.2?

Exercise 1.1.6. Compute

(1) $\langle x^2, \frac{1}{\sqrt{2}} \rangle$
(2) $\langle x^2, \sin(nx) \rangle$ for $n > 0$. *Hint:* $\sin(-x) = -\sin(x)$.
(3) $\langle x^2, \cos(nx) \rangle$ for $n > 0$. *Hint: integration by parts twice.*

Exercise 1.1.7. Combine the previous exercise and Fact 1.1.4 to obtain a Fourier series expansion for x^2. Then solve the Basel problem by evaluating at $x = \pi$.

Exercise 1.1.8. We now will verify that the functions from Exercise 1.1.3 are really orthonormal. Recall the complex number i is a square root of -1, i.e. $i^2 = -1$.

(1) Use the Taylor series for e^x, $\sin(x)$, and $\cos(x)$ to verify the identity
$$e^{ix} = \cos(x) + i\sin(x)$$
(2) Show
$$\int_{-\pi}^{\pi} e^{inx} = \begin{cases} 0 & \text{for } n \neq 0 \\ 2\pi & \text{for } n = 0. \end{cases}$$
Here to integrate a complex valued function you integrate the real and imaginary parts separately.
(3) The following identity holds still for complex numbers z_1 and z_2:
$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$
Use this to show that the functions $\frac{1}{\sqrt{2}}$, $\cos(nx)$ for $n > 0$, and $\sin(nx)$ for $n > 0$ are orthonormal for the inner product above as verified experimentally in Exercise 1.1.3.
Hint: For $m, n > 0$ write $e^{\pm imx}e^{\pm inx}$ in two different ways and use that $\cos(-y) = \cos(y)$, $\sin(-y) = -\sin(y)$ *to cancel out and get the terms you want in the real and imaginary places.*

1.2. **Challenge problem.** Remember this is supposed to be hard! In particular, don't feel bad if you spend hours thinking about one and don't solve it, but also don't let this warning stop you from trying!

Exercise (The density of squarefree integers and polynomials)**.** Compute "the probability that a randomly selected integer is squarefree." Here an integer is squarefree if any prime appears in its factorization at most once – for example, $35 = (5)(7)$ is squarefree, but $98 = (2)(7^2)$ is not.

(1) Find a way to make precise sense of the quantity in quotation marks.
(2) Come up with a heuristic answer. What does it have to do with the Basel problem?
(3) Check your guess numerically using CoCalc
(4) Turn the heuristic into a proof.
(5) What if we consider a random polynomial in $\mathbb{F}_p[x]$ instead of a random integer? Repeat the steps above, but ignore the Basel problem.