# HIDDEN STRUCTURE AND COMPUTATION - 2021 PRE-REU
# ABSTRACT VECTOR SPACES (2021-06-22)

## 1. Abstract vector spaces

**Definition 1.0.1.** A *vector space* $V$ over a field $\mathbb{K}$ (or, a $\mathbb{K}$-vector space) is a place where we can:

(1) Add two *vectors* $\vec{v}_1$ and $\vec{v}_2$ in $V$ to get a third vector, $\vec{v}_1 + \vec{v}_2$ in $V$.
(2) Multiply a vector $\vec{v} \in V$ by a *scalar* $\lambda \in \mathbb{K}$ to get a new vector $\lambda\vec{v} \in V$.

Addition should satisfy all the normal rules, and multiplication by a scalar distributes over addition of vectors

$$\lambda(\vec{v}_1 + \vec{v}_2) = \lambda\vec{v}_1 + \lambda\vec{v}_2.$$

In particular, we have a zero vector $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$ for any vector $\vec{v} \in V$. Moreover, we have, for any $\vec{v} \in V$,

(1) $1\vec{v} = \vec{v}$
(2) $0\vec{v} = \vec{0}$

**Example 1.0.2.**

(1) The set of height $n$ column vectors $\mathbb{K}^n$ is a $\mathbb{K}$-vector space.
(2) More generally, the set of all $m \times n$ matrices with entries in $\mathbb{K}$ is a vector space over $\mathbb{K}$. The column vectors $\mathbb{K}^n$ are the same as $n \times 1$ matrices over $\mathbb{K}$.
(3) The set $\mathbb{K}[x]$ of polynomials in the variable $x$ with coefficients in $\mathbb{K}$ is a $\mathbb{K}$-vector space.
(4) The set of infinite sequences $(a_1, a_2, a_3, \ldots)$ of real numbers is a vector space over $\mathbb{R}$:

$$(a_1, a_2, a_3, \ldots) + (a_1', a_2', a_3', \ldots) = (a_1 + a_1', a_2 + a_2', a_3 + a_3', \ldots),$$
$$\lambda(a_1, a_2, a_3, \ldots) = (\lambda a_1, \lambda a_2, \lambda a_3, \ldots).$$

(5) The set $C^0([0,1])$ of continuous real-valued functions on the closed interval $[0,1]$ is a vector space over $\mathbb{R}$:

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x).$$

(6) The field $\mathbb{C}$ is also a vector space over $\mathbb{R}$.
(7) The fields $\mathbb{C}$ and $\mathbb{R}$ are also vector spaces over $\mathbb{Q}$.

### 1.1. Subspaces.

**Definition 1.1.1.** If $V$ is a vector space and $W \subset V$, then $W$ is a *subspace* if $W$ contains the zero vector is closed under addition and scalar multiplication. A subspace of a vector space is itself a vector space!

**Example 1.1.2.**

(1) The set of all column vectors $\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}^t$ such $\sum_{i=1}^n a_i = 0$ is a subspace of $\mathbb{K}^n$.
(2) If $\vec{v}$ is a vector in $\mathbb{K}^n$, the set of all multiples $\lambda\vec{v}$ for $\lambda \in \mathbb{K}$ is a subspace.
(3) The set $\mathbb{K}[x]_{\leq n}$ of all polynomials with coefficients in $\mathbb{K}$ of degree at most $n$ is a subspace of the $\mathbb{K}$-vector space $\mathbb{K}[x]$
(4) If we view $\mathbb{C}$ and $\mathbb{R}$ as $\mathbb{Q}$-vector spaces, then $\mathbb{R}$ is a subspace of $\mathbb{C}$.

**Definition 1.1.3.** If $S$ is a set of vectors in a vector space $V$, the *span* of $S$, $\mathrm{Span}S$, is the set of all linear combinations of the vectors in $S$. It is a subspace of $V$.

**Example 1.1.4.** If $S = \{1, x^2, x^4, x^6, x^8, \dots\} \subset \mathbb{K}[x]$, then the span of $S$ is the subspace of polynomials with terms of only even degree.

**Definition 1.1.5.** A set of vectors $S$ is called *linearly dependent* if there are vectors $\vec{v}_1, \dots, \vec{v}_k \in S$ and scalars $a_1, \dots, a_k$ not all zero such that

$$a_1\vec{v}_1 + \dots + a_k\vec{v}_k = \vec{0}.$$

If no such combination exists, the vectors in $S$ are called *linearly independent*.

1.2. **Linear transformations.**

**Definition 1.2.1.** A map $T : V \to W$ between $\mathbb{K}$ vector spaces $V$ and $W$ is a *linear transformation* if

(1) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$.
(2) $T(\lambda\vec{v}) = \lambda T(\vec{v})$ for all $\vec{v} \in V$, $\lambda \in \mathbb{K}$.

**Example 1.2.2.** Suppose $f(x) \in \mathbb{R}[x]$. Then multiplication by $f$ is a linear transformation

$$T : \mathbb{R}[x] \to \mathbb{R}[x],\ g(x) \mapsto g(x)f(x).$$

**Definition 1.2.3.** For $T : V \to W$ a linear transformation,

(1) The *kernel* of $T$, written $\ker T$, is the set of all vectors $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. It is a subspace of $V$.
(2) The *image* of $T$, written Image$T$, is the set of

$$\{T(\vec{v}) \mid \vec{v} \in V\} \subset W.$$

It is a subspace of $W$.

**Definition 1.2.4.** If $V$ is a vector space and $T : V \to V$ is a linear transformation, a non-zero vector $\vec{v} \in V$ is an *eigenvector* for $T$ with *eigenvalue* $\lambda \in \mathbb{K}$ if $T(\vec{v}) = \lambda\vec{v}$. The eigenvalues of $T$ are the $\lambda \in \mathbb{K}$ such that $T$ admits an eigenvector with eigenvalue $\lambda$.

**Exercise 1.2.5.** Consider the left-shift operator $T$ on the vector space of sequences of real numbers

$$T(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots).$$

1. What are the eigenvalues of $T$? For each eigenvalue, what are the eigenvectors?
2. What are the eigenvalues of $T$ if I instead consider it as a linear transformation on the vector space of *convergent* sequences of real numbers?

**Exercise 1.2.6.** Let $C^\infty(\mathbb{R})$ be the vector space of infinitely differentiable real valued functions on the real line. We define a linear transformation $T$ of $C^\infty(\mathbb{R})$ by differentiation: $T(f) = \frac{df}{dx}$. What are the eigenvalues of $T$?

1.3. **Bases and dimension.**

**Definition 1.3.1.** A *basis* of a vector space $V$ is a linearly independent set of vectors $S$ in $V$ such that $\mathrm{Span}(S) = V$.

**Fact 1.3.2.** Any vector spaces admits a basis, and that any two bases can be put in bijection (this is a generalization to possibly infinite sets of having the same size)

**Definition 1.3.3.** The *dimension* of a $\mathbb{K}$-vector space $V$ is the cardinality of a basis. If you don't know what cardinality means, don't worry about it for now: We say that $V$ is *finite dimensional* if it has a basis consisting of finitely many vectors, and in that case, the dimension is the number of vectors in any basis. We write the dimension as $\dim V$ or $\dim_{\mathbb{K}} V$.

**Definition 1.3.4.** If $V$ is a finite dimensional vector space and $\vec{f}_1, \dots, \vec{f}_n$ is a basis (so $n = \dim V$), then we write

$$[\vec{v}]_{\vec{f}_\bullet} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ where } \vec{v} = a_1 \vec{f}_1 + \dots a_n \vec{f}_n.$$

**Definition 1.3.5.** If $T : V \to W$ is a map of finite dimensional vector spaces, $\vec{v}_1, \dots, \vec{v}_n$ is a basis of $V$, $\vec{w}_1, \dots, \vec{w}_n$ is a basis of $W$, we write

$$_{\vec{w}_\bullet}[T]_{\vec{v}_\bullet}$$

for the $m \times n$ matrix ($m = \dim W$, $n = \dim V$)

$$_{\vec{w}_\bullet}[T]_{\vec{v}_\bullet} = \begin{bmatrix} [T(\vec{v}_1)]_{\vec{w}_\bullet} & [T(\vec{v}_2)]_{\vec{w}_\bullet} & \dots & [T(\vec{v}_n)]_{\vec{w}_\bullet} \end{bmatrix}$$

If $T : V \to V$, then as before we write

$$[T]_{\vec{v}_\bullet} = {}_{\vec{v}_\bullet}[T]_{\vec{v}_\bullet}$$

**Fact 1.3.6.** For any $\vec{v} \in V$,

$$_{\vec{w}_\bullet}[T]_{\vec{v}_\bullet}[\vec{v}]_{\vec{v}_\bullet} = [T(\vec{v})]_{\vec{w}_\bullet}.$$

**Definition 1.3.7.** A linear transformation $T : V \to V$ diagonalizable if there is a basis of $V$ consisting of eigenvectors for $T$.

**Fact 1.3.8** (Rank + Nullity)**.** If $V$ and $W$ are finite dimensional vector spaces, then

$$\dim V = \dim \text{Image} T + \dim \ker T.$$

**Fact 1.3.9** (Ranks of matrices revisited)**.** For $A$ an $m \times n$ matrix with entries in a field $\mathbb{K}$, the following quantities are equal:

(1) The rank of $A$
(2) The number of leading ones in the reduced row echelon form of $A$
(3) The dimension of the span of the columns of $A$.
(4) The dimension of the span of the rows of $A$.
(5) $n - \dim \ker A$.

## 2. Problems

### 2.1. Vector spaces, subspaces, and linear transformations.

**Exercise 2.1.1.**

(1) Show that the set of polynomials with real coefficients that have a root at $x = 5$ is a subspace of $\mathbb{R}[x]$.
(2) Show that the set of all convergent sequences of real numbers is a subspace of the vector space of all sequences of real numbers.
(3) The set $C^1([0,1])$ of differentiable real-valued functions on the closed interval $[0,1]$ is a subspace of the real vector space $C^0([0,1])$.

**Exercise 2.1.2.** Consider the vector space $\mathbb{R}[x]_{<n}$ of polynomials of degree less than $n$. For any $a \in \mathbb{R}$, show that

$$1, (x-a), (x-a)^2, \dots, (x-a)^{n-1}$$

are a basis. Given $f(x) \in \mathbb{R}[x]_{\le n}$, use calculus to compute

$$[f(x)]_{\{1,(x-a),(x-a)^2,\dots,(x-a)^{n-1}\}}$$

**Exercise 2.1.3.**

(1) Show that $\sin(x)$ and $\cos(x)$ are linearly independent vectors in the $\mathbb{R}$-vector space of infinitely differentiable real-valued functions on the real line.
(2) Show that they are both eigenvectors with eigenvalue $-1$ for the linear transformation.

$$T(f) = \frac{d^2 f}{(dx)^2}$$

(3) The $-1$-*eigenspace* is the set of all eigenvectors with eigenvalue $-1$ plus the zero vector. Show this is a subspace, and that it is is preserved by the linear transformation $\frac{d}{dx}$.
(4) It can be shown that $\cos(x)$ and $\sin(X)$ are a basis for this *eigenspace*, that is, that any other eigenvector with eigenvalue $-1$ is of the form

$$a\cos(x) + b\sin(x)$$

for real numbers $a$ and $b$. Given this, compute the matrix of $\frac{d}{dx}$ in the basis $\sin(x), \cos(x)$.

**Exercise 2.1.4.** Find a basis for $\mathbb{C}$ as an $\mathbb{R}$-vector space.

**Exercise 2.1.5.**
1. Show that the elements 1 and $\sqrt{2}$ in $\mathbb{R}$, viewed as a $\mathbb{Q}$-vector space, are linearly independent.
2. $\star$ Show that 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$. Generalize this as far as you can.

**Exercise 2.1.6.** Show that $T(\vec{0}) = \vec{0}$ for any linear transformation $T$.

**Exercise 2.1.7.** Suppose $A$ is an $m \times n$ matrix with entries in $\mathbb{K}$. Then, we obtain a map

$$T : \mathbb{K}^n \to \mathbb{K}^m,\ T(\vec{v}) = A\vec{v}$$

(1) Show that $T$ is a linear transformation.
(2) Show that any linear transformation from $\mathbb{K}^n$ to $\mathbb{K}^m$ is obtained from a unique $m \times n$ matrix in this way.

**Exercise 2.1.8.** Consider the map

$$\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x].$$

given by differentiation.
(1) Show $\frac{d}{dx}$ is a linear transformation.
(2) What is the kernel of $\frac{d}{dx}$? What is its image?
(3) We can think of the vector space $\mathbb{R}[x]$ as being the space of infinite column vectors where all but finitely many entries are nonzero:

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \leftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \ldots \\ a_n \\ 0 \\ 0 \\ \ldots \end{bmatrix}.$$

Compute

$$\frac{d}{dx}\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \ldots \end{bmatrix}\right)$$

in this notation.

(4) Show that the space $\mathbb{R}[x]_{<n}$ of polynomials of degree $< n$ is a subspace of $\mathbb{R}[x]$. It is identified with $\mathbb{R}^n$ by the above recipe.
(5) Show that if $f \in \mathbb{R}[x]_{<n}$, then $\frac{d}{dx}^n(f) = 0$. (This notation means apply the linear transformation $\frac{d}{dx}$ iteratively $n$ times: first we get $\frac{d}{dx}(f)$, then $\frac{d}{dx}\left(\frac{d}{dx}(f)\right)$, and so on).

**Exercise 2.1.9.** Let $V$ be the vector space of all convergent infinite sequences of real numbers.

(1) Show that the map
$$\lim : (a_0, a_1, \dots) \mapsto \lim_{n\to\infty} a_n$$
is a linear transformation from $V$ to $\mathbb{R}$ (here we think of $\mathbb{R}$ as a vector space over itself).
(2) Show that for any $\vec{v} \in V$, there is a unique expression
$$\vec{v} = \vec{s} + \vec{c}$$
where $\vec{s}$ is in the kernel of lim and $\vec{t}$ is a constant sequence (that is $\vec{c}$ is of the form $(a, a, a, \dots)$ for some real number $a$).

**Exercise 2.1.10.** Let $V$ be the vector space of infinite sequences of real numbers
$$(a_0, a_1, a_2, \dots)$$
We have the left and right shift operators:
$$L((a_0, a_1, a_2, \dots)) = (a_1, a_2, \dots) \text{ and } R((a_0, a_1, a_2, \dots)) = (0, a_0, a_1, \dots).$$

(1) What are the eigenvalues and eigenvectors of $L$ and $R$?
(2) What are the eigenvalues and eigenvectors if we restrict $L$ and $R$ to the subspace of convergent sequences?

**Exercise 2.1.11.** If $A$ is an $m \times n$ matrix, explain why $\dim \ker A$ is the number of free variables in the system of equations
$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}.$$

**Exercise 2.1.12.** If $V_1$ and $V_2$ are two vector spaces over $\mathbb{K}$, write $\mathrm{Hom}(V_1, V_2)$ for the set of all linear transformations from $V_1$ to $V_2$. Show that $\mathrm{Hom}(V_1, V_2)$ is a vector space over $\mathbb{K}$ (how are addition and scalar multiplication defined?). If $V_1 = \mathbb{K}^n$ and $V_2 = \mathbb{K}^m$, give a description of this vector space using objects we have already studied.

**Exercise 2.1.13.** For $T : V \to W$ a linear transformation, $\vec{v}_\bullet, \vec{v}'_\bullet$ two bases of $V$, and $\vec{w}_\bullet, \vec{w}'_\bullet$ two bases of $W$:

**A.** Explain why
$${}_{\vec{w}'_\bullet}[T]_{\vec{v}'_\bullet} = {}_{\vec{w}'_\bullet}[\mathrm{Id}_W]_{\vec{w}_\bullet}\, {}_{\vec{w}_\bullet}[T]_{\vec{v}_\bullet}\, {}_{\vec{v}_\bullet}[\mathrm{Id}_V]_{\vec{v}'_\bullet}.$$
**B.** Compare this to the change of basis formula discussed previously for $T : \mathbb{K}^n \to \mathbb{K}^n$.

## 2.2. The Eventown theorem.

**Exercise 2.2.1.** Justify Fact 1.3.8 using the following steps:

**A.** Show that if $\vec{v}_1, \dots, \vec{v}_k$ are a basis for $\ker T$ and $\vec{w}_1, \dots, \vec{w}_\ell$ are vectors in $V$ such that $T(\vec{w}_1, \dots, T(\vec{w}_\ell)$ are basis for Image$T$ then together
$$\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \vec{w}_\ell$$
form a basis for $V$.
*Hint: Show separately that they span $V$ and that they are linearly independent.*

**B.** Conclude.

**Exercise 2.2.2.** Justify Fact 1.3.9. *Hint: The span of the columns of a matrix $A$ is equal to the image of the corresponding linear transformation, so you can use Fact 1.3.8.*

**Exercise 2.2.3.** If $S \subset \mathbb{K}^n$ is a subset, we write $S^\perp$ for the set of vectors $\vec{v} \in \mathbb{K}^n$ such that $\vec{s} \cdot \vec{v} = 0$ for every $\vec{s} \in S$. A set of vectors $S \in \mathbb{K}^n$ is *isotropic* if $S \subset S^\perp$.

**A.** Show that $S^\perp$ is a subspace of $\mathbb{K}^n$.
**B.** Show that $S^\perp = (\mathrm{Span} S)^\perp$.
**C.** Show that if $S$ is an isotropic set of vectors in $\mathbb{K}^n$ then $\mathrm{Span} S$ is an isotropic subspace.
**D.** If $S$ is a subspace of $\mathbb{K}^n$, show that $\dim S + \dim S^\perp = n$.
*Hint: Construct a $\dim W \times n$ matrix whose rank is $\dim W$ and whose kernel is $W^\perp$, then apply rank+nullity.*
**E.** Show that an isotropic subspace of $\mathbb{K}^n$ has dimension at most $\lfloor n/2 \rfloor$.
**F.** $\star$ Show that any isotropic subspace of $\mathbb{F}_2^n$ is contained in an isotropic subspace of dimension $\lfloor n/2 \rfloor$.

**Exercise 2.2.4.** Recall the rules of Eventown: every club has an even number of members, and any two clubs share an even number of members. We will now prove:

**Theorem** (The Eventown Theorem)**.** *If Eventown has $n$ residents, then any maximal system of clubs consists of exactly $2^{\lfloor n/2 \rfloor}$ clubs.*

**A.** Show that the membership vectors of a valid set of clubs in Eventown is an isotropic set in $\mathbb{F}_2^n$ and vice versa.
**B.** Conclude using the results of Exercise 2.2.3.