THE SPECTRAL p-ADIC JACQUET-LANGLANDS CORRESPONDENCE AND A QUESTION OF SERRE

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Abstract. We show that the completed Hecke algebra of p-adic modular forms is isomorphic to the completed Hecke algebra of continuous p-adic automorphic forms for the units of the quaternion algebra ramified at p and ∞. This gives an affirmative answer to a question posed by Serre in a 1987 letter to Tate. The proof is geometric, and lifts a mod p argument due to Serre: we evaluate modular forms by identifying a quaternionic double-coset with a fiber of the Hodge-Tate period map, and extend functions off of the double-coset using fake Hasse invariants. In particular, this gives a new proof, independent of the classical Jacquet-Langlands correspondence, that Galois representations can be attached to classical and p-adic quaternionic eigenforms.

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1. Introduction

Let p be a prime, and let $D/\mathbb{Q}$ be the quaternion algebra ramified at p and ∞. Let $\mathbb{A}$ denote the adèles of $\mathbb{Q}$, $\mathbb{A}_f$ the finite adèles, and $\mathbb{A}_f^{(p)}$ the finite prime-to-p adèles. Let $K^p \subset D^\times(\mathbb{A}_f^{(p)})$ be a compact open subgroup. For R a topological ring (e.g. $\mathbb{C}$, $\overline{\mathbb{F}_p}$, $\mathbb{Q}_p$, or $\mathbb{C}_p$), we consider the space of continuous p-adic automorphic forms on $D^\times$ with coefficients in R and prime-to-p level K^p,

$$\mathcal{A}_R^{K^p} := \mathrm{Cont}(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})/K^p, R).$$

For R totally disconnected (e.g. $\overline{\mathbb{F}_p}$, $\mathbb{Q}_p$, or $\mathbb{C}_p$), the archimedean component can be removed, and we have an identification

$$\mathcal{A}_R^{K^p} = \mathrm{Cont}(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K^p, R).$$

Note that $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K^p$ is a profinite set. Moreover, by choosing coset representatives, it can be identified with a finite disjoint union of compact open subgroups of $D^\times(\mathbb{Q}_p)$, so that it is essentially a p-adic object.

The space $\mathcal{A}_R^{K^p}$ admits an action of the abstract double-coset Hecke algebra

$$\mathbb{T}_{\mathrm{abs}} := \mathbb{Z}[K^p\backslash D^\times(\mathbb{A}_f^{(p)})/K^p],$$

and a commuting action of $D^\times(\mathbb{Q}_p)$. In this work we, we study the spectral decomposition of $\mathcal{A}_R^{K_p}$ under the action of $\mathbb{T}_{\mathrm{abs}}$.

The classical Jacquet-Langlands correspondence [15], proved using analytic techniques, implies that, up to twisting, the eigensystems for $\mathbb{T}_{\mathrm{abs}}$ acting on $\mathcal{A}_{\mathbb{C}}^{K^p}$ are a strict subset of those appearing in classical complex modular forms (on the quaternionic side, one only sees the eigensystem attached to a cuspidal modular form if the corresponding automorphic representation of GL_2 is discrete series at p).

On the other hand, arguing with the geometry of mod p modular curves, Serre [24] showed that the eigensystems arising in $\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}$ are the *same* as those appearing in the space of mod p modular forms (cf. Theorem 1.1.1 below for a slight refinement of Serre's result). In particular, the gaps in the Jacquet-Langlands correspondence over $\mathbb{C}$ disappear when working mod p.

The main result of this work is a natural lift of Serre's result to $\mathbb{Q}_p$: we use the geometry of the perfectoid modular curve at infinite level to show that the completed Hecke algebra of $\mathcal{A}_{\mathbb{Q}_p}^{K^p}$ is isomorphic to the completed Hecke algebra of p-adic modular forms (cf. Theorem A below for a precise statement).

Theorem A is compatible with the classical Jacquet-Langlands correspondence: the eigensystems appearing in classical complex quaternionic automorphic forms can be identified with the eigensystems appearing in $\mathcal{A}_{\mathbb{Q}_p}^{K_p}$ such that the corresponding eigenspace contains a vector which, up to a twist, transforms via an algebraic representation of $D^\times(\mathbb{Q}_p)$ after restriction to a sufficiently small compact open subgroup. Thus, Theorem A can be interpreted as saying that there is a p-adic Jacquet-Langlands correspondence that fills in the gaps in the classical Jacquet-Langlands correspondence. As our proof of the p-adic correspondence is independent of the classical correspondence, we also obtain a new proof that Galois representations can be attached to quaternionic automorphic forms (cf. Corollary B below for a precise statement).

Both Theorems 1.1.1 and A are purely *spectral* Jacquet-Langlands correspondences – in the present article we do not make any attempt to describe the $D^\times(\mathbb{Q}_p)$-representation appearing in an eigenspace. Nevertheless, the methods employed in the proofs of Theorem 1.1.1 and A can be used provide significant information about these local representations, and we return to this in another work [12] (cf. 1.4 below for further discussion).

1.1. **Serre's spectral mod p Jacquet-Langlands correspondence.** Before discussing our results and techniques further, we take a brief detour to recall a precise statement of Serre's [24] mod p correspondence.

If we fix an isomorphism

$$D^\times(\mathbb{A}_f^{(p)}) \cong \mathrm{GL}_2\left(\mathbb{A}_f^{(p)}\right),$$

then we obtain an action of the Hecke algebra $\mathbb{T}_{\mathrm{abs}}$ on the space $M_{\mathbb{F}_p}^{K^p}$ of mod p modular forms of prime-to-p level K^p. In a 1987 letter to Tate, Serre [24] proved a mod p Jacquet-Langlands correspondence comparing the spectral decompositions of $\mathcal{A}_{\mathbb{F}_p}^{K^p}$ and $M_{\mathbb{F}_p}^{K^p}$. We state below a slight strengthening of his result, which follows from essentially the same proof given in loc. cit. First, some notation:

Suppose $\mathbb{T}' \subset \mathbb{T}_{\mathrm{abs}} \otimes \overline{\mathbb{F}_p}$ is a commutative sub-algebra and $\chi : \mathbb{T}' \to \overline{\mathbb{F}_p}$ is a character. Then, if $\mathbb{T}'$ acts on an $\overline{\mathbb{F}_p}$-vector space V, we may consider the χ-eigenspace $V[\chi]$. If we write $\mathfrak{m}_\chi := \ker \chi$, we may also consider the generalized χ-eigenspace $V_{\mathfrak{m}_\chi}$ (that is, the subset of elements killed by a power of $\mathfrak{m}_\chi$).

Theorem 1.1.1 (Serre)**.** *Let $\mathbb{T}' \subset \mathbb{T}_{\mathrm{abs}}$ be a commutative sub-algebra. Then, there is a finite collection of characters $\chi_i : \mathbb{T}' \to \overline{\mathbb{F}_p}$ with kernels $\mathfrak{m}_i$ such that:*

(1) *For each i, $\left(\mathcal{A}_{\mathbb{F}_p}^{K^p}\right)_{\mathfrak{m}_i}$ and $\left(M_{\mathbb{F}_p}^{K^p}\right)_{\mathfrak{m}_i}$ are non-zero; in particular,*
$$\mathcal{A}_{\mathbb{F}_p}^{K^p}[\chi_i] \neq 0 \text{ and } M_{\mathbb{F}_p}^{K^p}[\chi_i] \neq 0, \text{ and}$$
(2) *there are direct sum decompositions*
$$\mathcal{A}_{\mathbb{F}_p}^{K^p} = \bigoplus_i \left(\mathcal{A}_{\mathbb{F}_p}^{K^p}\right)_{\mathfrak{m}_{\chi_i}} \text{ and } M_{\mathbb{F}_p}^{K^p} = \bigoplus_i \left(M_{\mathbb{F}_p}^{K^p}\right)_{\mathfrak{m}_{\chi_i}}.$$

In other words, the Hecke eigensystems appearing in mod p quaternionic automorphic forms are precisely those appearing in mod p modular forms. This stands in contrast to the classical Jacquet-Langlands correspondence, where the eigensystems appearing in quaternionic forms are a strict subset of those appearing in modular forms. The following example gives a concrete illustration:

Example 1.1.2. The discriminant form, represented by the Ramanujan series

$$\Delta(q) = q \prod_{n \geq} (1 - q^n)^{24} = \sum \tau(n) q^n,$$

is a weight 12 cuspidal eigenform whose corresponding automorphic representation is principal series at every prime p. Thus, the corresponding Hecke eigensystem, encoded by the coefficients $\tau(\ell)$ for ℓ prime, does not appear in the space of classical automorphic forms on $D^\times$ for any prime p (recall p appears in the definition of $D^\times$).

By contrast, Theorem 1.1.1 shows that the coefficients $\tau(\ell) \mod p$ for $\ell \neq p$ are remembered by a mod p quaternionic automorphic form on $D^\times$. A similar phenomena occurs in our p-adic correspondence, which remembers the numbers $\tau(\ell)$ on the nose!

1.2. A spectral p-adic Jacquet-Langlands correspondence.

1.2.1. *Serre's question.* Serre ended his letter to Tate with a list of questions inspired by the mod p Jacquet-Langlands correspondence. One of these (cf. [24, paragraph (26)]), transcribed below, suggests an analogous study relating $\mathcal{A}_{\mathbb{Q}_p}^{K^p}$ to p-adic modular forms.

> (26) *Analogues p-adiques.* Au lieu de regarder les fonctions localement constantes sur $D_{\mathbb{A}}^\times/D_{\mathbb{Q}}^\times$ á valeurs dans $\mathbb{C}$, il serait plus amusant de regarder celles à valeurs dans $\overline{\mathbb{Q}}_p$. Si l'on décompose $\mathbb{A}$ en $\mathbb{Q}_p \times \mathbb{A}'$, on leur imposerait d'être localement constantes par rapport à la variable dans $D_{\mathbb{A}'}$ et d'être continues (ou analytiques, ou davantage) par rapport à la variable dans D_p... Y aurait-il des représentations galoisiennes p-adiques associées a de telles fonctions, supposées fonctions propres des opérateurs de Hecke? Peut-on interpréter les constructions de Hida (et Mazur) dans un tel style? Je n'en ai aucune idée.

Our main result, Theorem A below, shows that the answers to Serre's questions are, largely, yes. In particular, Theorem A implies that Galois representations can be attached to p-adic quaternionic eigenforms (Corollary B below).

1.2.2. *Completed actions.* Before stating Theorem A, we introduce some notation for discussing operators on p-adic Banach spaces. Let K be a complete extension of $\mathbb{Q}_p$ equipped with an absolute value $|\cdot|$ extending the p-adic absolute value on $\mathbb{Q}_p$. A Banach space over K is a K-vector space complete with respect to a norm $||\cdot||$ satisfying the ultrametric inequality and such that $||av|| = |a|\cdot||v||$ for all $a \in K$, $v \in V$. A linear operator $f : V \to V$ is bounded if there exists $C > 0$ such that $||f(v)|| \leq C||v||$ for all $v \in V$. A linear operator is continuous for the norm topology if and only if it is bounded. The operator norm of a bounded operator f is given by

$$||f|| = \sup_{v\neq 0} ||f(v)||/||v||.$$

We say that a bounded operator f is uniform if $||f|| \leq 1$ (this last piece of terminology is non-standard).

If A is an algebra acting on a K-Banach space V by linear transformations we say that the action is uniform if each $a \in A$ acts by a uniform operator on V. If A acts uniformly on V then we write A_V for the image of A in $\mathrm{End}_{\mathrm{cont}}(V)$, and $A_V^\wedge$ for the closure of A_V for the topology of pointwise convergence. Uniformity implies that $A_V^\wedge$ is a sub-algebra of $\mathrm{End}_{\mathrm{cont}}(V)$.

1.2.3. *A homeomorphism of completed Hecke algebras.* The space $\mathcal{A}_{\mathbb{Q}_p}^{K^p}$ of p-adic quaternionic automorphic forms is a $\mathbb{Q}_p$-Banach space with respect to the sup norm, and the action of $\mathbb{T}_{\mathrm{abs}}$ is uniform with respect to this norm. We can thus form the completed Hecke algebra $\mathbb{T}'^\wedge_{\mathcal{A}_{\mathbb{Q}_p}^{K^p}}$.

As in the mod p case, we would like to compare the Hecke action on $\mathcal{A}_{\mathbb{Q}_p}^{K^p}$ to a Hecke action on a space of modular forms; in this case, we will do so by comparing completed Hecke algebras. To that end: Serre [26] constructed natural spaces of

p-adic modular forms by completing spaces of classical modular forms for the p-adic topology on q-expansions (these spaces were then interpreted geometrically by Katz [19]). In particular, one obtains a natural $\mathbb{Q}_p$-Banach space $M^{K^p}_{p-\mathrm{adic}}$ of p-adic modular forms of prime-to-p level K^p equipped with a uniform action of $\mathbb{T}_{\mathrm{abs}}$.

Theorem A. *For any sub-algebra $\mathbb{T}' \subset \mathbb{T}_{\mathrm{abs}}$, the identity map $\mathbb{T}' \to \mathbb{T}'$ extends to a canonical homeomorphism of topological $\mathbb{Z}_p$-algebras*

$$\mathbb{T}'^{\wedge}_{\mathcal{A}^{K^p}_{\mathbb{Q}_p}} = \mathbb{T}'^{\wedge}_{M^{K^p}_{p-\mathrm{adic}}}.$$

Theorem A implies Theorem 1.1.1 above, and gives a natural lift to characteristic zero suitable, e.g., for the construction of Galois representations. Moreover, our proof is in some sense a lift of Serre's proof via the mod p geometry of modular curves to a proof in characteristic zero via the p-adic geometry of infinite level modular curves.

Remark 1.2.4. The completed Hecke algebras do not change if we replace $\mathbb{Q}_p$ with a finite extension, or even $\mathbb{C}_p$ (this invariance under base change plays an important role in our proof). Thus, although Serre in his letter quoted above suggests the study of $\mathcal{A}^{K^p}_{\overline{\mathbb{Q}_p}}$, it is natural in our setup to work over $\mathbb{Q}_p$. In particular, an eigenform in $\mathcal{A}^{K^p}_{\overline{\mathbb{Q}_p}}$ will still give rise to a $\overline{\mathbb{Q}_p}$-valued character of $\mathbb{T}'^{\wedge}_{\mathcal{A}^{K^p}_{\mathbb{Q}_p}}$.

1.2.5. *Completed cohomology.* By a result of Emerton [6] (building on work of Hida), $\mathbb{T}'^{\wedge}_{M^{K^p}_{p-\mathrm{adic}}}$ is equal to the completed Hecke algebra of $\mathbb{T}'$ acting on the completed cohomology of modular curves. On the other hand, $\mathcal{A}^{K^p}_{\mathbb{Q}_p}$ *is* the completed cohomology at level K^p for $D^\times$. Hence we also obtain a homeomorphism between the completed Hecke algebras for the completed cohomology of GL_2 and $D^\times$.

1.2.6. *Galois representations.* Let $\mathbb{T}_{\mathrm{tame}} \subset \mathbb{T}_{\mathrm{abs}}$ be the tame Hecke algebra of level K^p, i.e., the commutative sub-algebra generated by the Hecke operators at ℓ for primes $\ell \neq p$ at which K^p factors as $K^{p,\ell}K_\ell$ for $K^{p,\ell} \subset D^\times(\mathbb{A}_f^{(pl)})$ and $K_\ell \subset D^\times(\mathbb{Q}_l)$ a maximal compact subgroup. For each such ℓ we write T_ℓ for the standard unramified Hecke operator.

Using the well-known corresponding fact for $\mathbb{T}'^{\wedge}_{M^{K^p}_{p-\mathrm{adic}}}$, we deduce

Corollary B. *If $\chi : \mathbb{T}^{\wedge}_{\mathrm{tame},\mathcal{A}^{K^p}_{\mathbb{Q}_p}} \to \mathbb{C}_p$ is a continuous character then there exists a unique semisimple continuous representation*

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C}_p)$$

unramified at ℓ as above and such that $\mathrm{Tr}(\rho(\mathrm{Frob}_\ell)) = \chi(T_\ell)$.

One can obtain such a χ from a quaternionic eigenform as in Remark 1.2.4, and thus Corollary B attaches Galois representations to these eigenforms.

1.2.7. *Work of Emerton.* Corollary B can also be deduced from the classical Jacquet-Langlands correspondence. In fact, a version of Theorem A after localizing at a maximal idea was first shown by Emerton [7, 3.3.2] by reversing this argument: the classical correspondence gives rise to a surjective map of completed Hecke algebras

$$\mathbb{T}'^{\wedge}_{M^{K^p}_{p-\mathrm{adic}}} \to \mathbb{T}'^{\wedge}_{\mathcal{A}^{K^p}_{\mathbb{Q}_p}}$$

(which is enough to obtain Corollary B), and then strong results in the deformation theory of Galois representations can be used to deduce that this map is an isomorphism after localizing at a maximal ideal $\mathfrak{m}$ (under minor hypotheses on $\mathfrak{m}$).

By contrast, our proof is based entirely on the p-adic geometry of modular curves. Thus, we obtain a new proof of Corollary B that is independent of the classical Jacquet-Langlands correspondence, and our proof of Theorem A does not use any $R = \mathbb{T}$ theorems or other difficult results about Galois deformations.

1.3. **Sketch of proof.**

1.3.1. *Proof of Theorem 1.1.1.* Serre's proof of the mod p correspondence proceeds in two steps: First, one gives a Hecke-equivariant identification of the super-singular locus with a quaternionic double coset in order to transfer Hecke eigensystems from modular forms to the quaternion algebra. The evaluation map is not injective, but, dividing by a suitable power of the Hasse invariant ensures that every eigensystem can be transferred. That every quaternionic eigensystem can be recovered in this way follows from ampleness of the modular sheaf.

1.3.2. *Proof of Theorem A.* To prove the p-adic correspondence, we first give a Hecke-equivariant identification of a super-singular fiber of the Hodge-Tate period map on the infinite level modular curve with a quaternionic double-coset. This allows us to evaluate classical modular forms and transfer Hecke eigensystems from modular forms to the quaternion algebra. In this case, the evaluation map from classical modular forms is injective, but it is not surjective. Nonetheless, ampleness of the modular sheaf combined with Scholze's technique of fake Hasse invariants can be used to show the image is dense, which is enough to establish the result.

1.3.3. *The supersingular Igusa variety.* The relation between these proofs is clarified by highlighting the important role of the supersingular Igusa variety[1], that is, the moduli space of supersingular elliptic curves equipped with a trivialization of their formal group. After some choices, this Igusa variety can be identified with a quaternionic coset (which, as a profinite set, admits a natural scheme structure over $\overline{\mathbb{F}_p}$). Thus functions on the Igusa variety *are* mod p quaternionic automorphic forms. Moreover, the Igusa variety admits a canonical lift to a p-adic formal scheme (whose generic fiber over $\mathbb{C}_p$ is the simplest type of perfectoid space – a profinite set), and p-adic quaternionic forms are functions on this lift.

The mod p supersingular locus admits a punctual uniformization by the supersingular Igusa variety, and this uniformization allows us to transfer classical modular forms mod p to mod p quaternionic forms. As observed by Caraiani-Scholze [2], this punctual uniformization mod p lifts via Serre-Tate theory to a punctual uniformization of a fiber of the Hodge-Tate period map, allowing us to transfer classical modular forms to p-adic quaternionic modular forms. This uniformization and resulting transfer depend on the auxiliary choice of a lift of a formal group and a trivialization of its Tate module, and these choices can be made so that the p-adic transfer is literally a lift of the modulo p transfer.

As the uniformization maps in both the mod p and p-adic setting are defined using a moduli interpretation, they are well-behaved with respect to Hecke operators; on the other hand, since their images have simple geometric interpretations, one

[1]The role of this space is a bit obscured in Serre's presentation [24], which may partially explain why a p-adic generalization did not appear earlier.

has enough geometric tools available to prove that the induced map of completed Hecke algebras is an isomorphism.

1.4. **Eigenspaces and the local p-adic Jacquet-Langlands correspondence.** One failing of Theorem A as stated is that it says nothing about the eigenspace in quaternionic automorphic forms attached to a character of $\mathbb{T}'^{\wedge}_{\mathcal{A}^{K^p}_{\mathbb{Q}_p}}$ valued in an extension of $\mathbb{Q}_p$. Indeed, because completed Hecke algebras are formed by compiling congruences of eigensystems, which may not be reflected in congruences of eigenvectors, one does not even know whether such an eigenspace is non-empty. On the other hand, one expects that it is always non-empty, and that the $D^\times(\mathbb{Q}_p)$ representation appearing is essentially that constructed in the local correspondences of Knight [20] and Scholze [22].

Using the evaluation map from the proof of Theorem A, one can show this eigenspace is non-empty at least for eigensystems attached to classical (or even overconvergent) modular forms. Moreover, using Gross' [9] theory of algebraic automorphic forms and the clasical Jacquet-Langlands correspondence, one can completely describe the locally algebraic vectors, and show that our construction produces vectors which are not locally algebraic. Some results of this nature are discussed in the author's thesis [14] and were included in an earlier draft of the present work.

In fact, these results can be considerably improved by replacing the punctual uniformization used in the proofs of Theorems 1.1.1 and A with the full formal/p-adic uniformization of the supersingular locus. From this perspective, one obtains a natural set of model representations from distributions on Lubin-Tate space. A classical or overconvergent eigenform gives rise to a nontrivial map from a model representation to the corresponding eigenspace in quaternionic automorphic forms; the transfer maps considered in the present work are obtained by evaluating these more general maps at certain dirac distributions. This construction and related results will appear in [12].

1.5. **Outline.** In Section 2 we prove the version of Serre's mod p Jacquet-Langlands correspondence stated above as Theorem 1.1.1. The proof we give is essentially that of Serre, but we emphasize the role of the supersingular Igusa variety and uniformization. In particular, we introduce here the supersingular Igusa variety and carefully explain its connection with quaternionic cosets; this will play a key role also in the proof of Theorem A.

In Section 3 we prove Theorem A. After recalling some results about perfectoid modular curves and the Hodge-Tate period map from [21, 2], we construct our p-adic transfer map and show it is injective (a simple argument using that modular curves are one-dimensional). The other key property we need is density of the image, which we establish with the help of Scholze's fake Hasse invariants. We note that this density argument has been considerably simplified compared to the original version appearing in [14]; this simplification is made possible by the recent result "Zariski closed = strongly Zariski closed" of Bhatt-Scholze [1]. These properties of the evaluation map established, it remains only to prove some general lemmata for comparing completed Hecke algebras in order to deduce Theorem A.

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1.7. **Notation and conventions.** We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ and write $\mathbb{C}_p$ for its completion. We denote by $\breve{\mathbb{Q}}_p$ the completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$, and by $\breve{\mathbb{Z}}_p \subset \breve{\mathbb{Q}}_p$ the completion of the ring of integers in the maximal unramified extension. We write $\overline{\mathbb{F}_p}$ for $\breve{\mathbb{Z}}_p/p$, an algebraically closed extension of $\mathbb{F}_p = \mathbb{Z}_p/p$; note that we have a canonical identification $W(\overline{\mathbb{F}_p}) = \breve{\mathbb{Z}}_p$.

Give a topological space T, we denote by $\underline{T}$ the constant sheaf with value T, that is, the functor on schemes

$$\underline{T}(S) = \mathrm{Cont}(|S|, T)$$

where $|S|$ denotes the underlying topological space of S. When T is a profinite set, it is represented by $\mathrm{SpecCont}(T, \mathbb{Z})$, where $\mathbb{Z}$ is equipped with the discrete topology (so that the continuous functions are just locally constant).

2. The supersingular Igusa variety and the mod p correspondence

In this section we give a proof of Theorem 1.1.1, roughly following Serre [24]. The main difference between our presentation and that of loc. cit. is that here we emphasize from the beginning the role of the (big, supersingular) Igusa variety in the uniformization of the supersingular locus. In particular, we give a careful proof of the identification between the Igusa variety and the profinite set $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)$, which will also be needed for our p-adic results.

2.1. **Mod p modular curves.** For $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ a sufficiently small compact open subgroup, we write Y_{K^p} for the modular curve over $\overline{\mathbb{F}_p}$ parameterizing elliptic curves up to prime-to-p isogeny with K^p-level structure. We write X_{K^p} for its compactification and

$$Y = \lim Y_{K^p}, X = \lim X_{K^p}.$$

The modular curve Y parameterizes pairs $(E, \varphi_{\mathbb{A}_f^{(p)}})$, where E is an elliptic curve up to prime-to-p isogeny and

$$\varphi_{\mathbb{A}_f^{(p)}} : \underline{\mathbb{A}_f^{(p)}}^2 \xrightarrow{\sim} \left(\lim_{(p,n)=1} E[n] \right) \otimes \underline{\mathbb{Q}}$$

is a trivialization of the prime-to-p adelic Tate module of E. The group $\mathrm{GL}_2\left(\mathbb{A}_f^{(p)}\right)$ acts on Y by composition with $\varphi_{\mathbb{A}_f^{(p)}}$.

2.1.1. *Modular forms.* There is a universal elliptic curve up to prime-to-p-isogeny $\pi : E \to Y$. We consider the modular sheaf $\omega := \pi_* \Omega_{E/Y}$, which is naturally $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant. The $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$ action on Y extends uniquely to an action on X, and we extend ω equivariantly to X by taking sections with holomorphic q-expansions over cusps.

For each k, $H^0(X, \omega^k)$ is an admissible $\overline{\mathbb{F}_p}$ representation of $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$, and for $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ a sufficiently small compact open subgroup as above,

$$H^0(X, \omega^k)^{K^p} = H^0(X_{K^p}, \omega^k).$$

2.1.2. *The Hasse invariant.* The Hasse invariant, $H \in H^0(X, \omega^{p-1})$, is a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant section with vanishing locus equal to the supersingular locus X^{ss}. The supersingular locus is contained in Y, and we write $Y^{\mathrm{ss}} = X^{\mathrm{ss}}$, $X^{\mathrm{ord}} = X \backslash X^{\mathrm{ss}}$, $Y^{\mathrm{ord}} = Y \backslash Y^{\mathrm{ss}}$.

2.1.3. *Hecke algebras and generalized eigenspaces.* If L is a field and V is a finite dimensional vector space over L, then for any commutative L-algebra T acting on V, we have a decomposition into generalized eigenspaces

$$V = \bigoplus V_{\mathfrak{m}}$$

where $\mathfrak{m}$ runs over maximal ideals of T with residue field a finite extension of K and $V_{\mathfrak{m}}$ is the sub-module of $\mathfrak{m}$-torsion elements. In particular, if $V_{\mathfrak{m}} \neq 0$, the eigenspace $V[\mathfrak{m}]$ consisting of elements annihilated by $\mathfrak{m}$ is non-empty.

The same result applies more generally to V a colimit of finite dimensional vector spaces with T-action. In particular, if W is a colimit of admissible representations of a locally profinite group G, K is a compact open subset of G, and T is a commutative subalgebra of the abstract Hecke algebra $L[K \backslash G / K]$, then the action of T on the invariants W^K admits a decomposition into generalized eigenspaces.

This formalism gives a decomposition of $H^0(X_{K^p}, \omega^k)$ into generalized eigenspaces for the action of any commutative subalgebra

$$\mathbb{T}' \subset \overline{\mathbb{F}_p}[K^p \backslash \mathrm{GL}_2(\mathbb{A}_f^{(p)} / K^p] = \mathbb{T}^{\mathrm{abs}}_{K^p} \otimes \overline{\mathbb{F}_p},$$

and similarly for other related spaces to be introduced below.

2.2. **Mod p modular forms.** The ring of mod p modular forms is the $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant $\mathbb{Z}/(p-1)\mathbb{Z}$-graded ring

$$M_{\overline{\mathbb{F}_p}} := \bigoplus_{k \in \mathbb{Z}/(p-1)\mathbb{Z}} H^0(X^{\mathrm{ord}}, \omega^k) \cong \left(\bigoplus_{k \geq 0} H^0(X, \omega^k) \right) / (H-1)$$

where here to give a multiplication on the left-hand side we use H to identify ω^k with $\omega^{k+(p-1)}$ over X^{ord}; the two sides are isomorphic because multiplying by a sufficiently large power of H kills any poles along the supersingular locus.

Remark 2.2.1. To obtain a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-action inducing the standard action of Hecke operators on K^p-invariants, one should consider sections of $\Omega_X(\mathrm{cusps}) \otimes \omega^{k-2}$ rather than ω^k. The Kodaira-Spencer isomorphism gives an identification

$$\Omega_X(\mathrm{cusps}) \cong \omega^2,$$

however, the natural $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-actions differ through a twist by the unramified determinant character (because Serre duality, which is not isogeny invariant, is used in the definition of the isomorphism). This serves only to replace each Hecke operator with a multiple by an invertible element (since we do not include Hecke operators at p!), so it will not change the resulting Hecke algebra.

We prefer the action used above because if we equip $M_{\overline{\mathbb{F}_p}}$ with the twisted action then the ring multiplication is no longer equivariant. Indeed, it is most natural to consider the space of mod p modular forms with the $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-action that gives rise to the standard Hecke action as a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant graded free module of rank 1 over the ring of mod p modular forms we have defined here.

Remark 2.2.2. There are at least two other natural constructions of $M_{\overline{\mathbb{F}_p}}$:

(1) It is the span of the images of $H^0(X, \omega^k)$ over all k under the total q-expansion map (we emphasize that one must take at least one cusp in each connected component). Mod p modular forms were first studied in this context, e.g. by Serre [26].
(2) It is the space of functions on the first level of the ordinary Igusa tower $\mathrm{Ig}_1^{\mathrm{ord}}$ over X_{ord} (cf. e.g. [8]); this is a $(\mathbb{Z}/p\mathbb{Z})^\times$-torsor whose restriction to Y_{ord} parameterizes trivializations $\mu_p \xrightarrow{\sim} \widehat{E}[p]$. The $\mathbb{Z}/(p-1)\mathbb{Z}$ grading is just the decomposition according to characters of $(\mathbb{Z}/p\mathbb{Z})^\times$, and modular forms are embedded as functions by evaluation along the pullback of $\frac{dt}{t}$ to $\omega_{\widehat{E}[p]} = \omega_E$.

The latter perspective is the same that we use over the supersingular locus to relate modular forms to quaternionic automorphic forms.

2.2.3. *Spectral decomposition of $M_{\overline{\mathbb{F}_p}}$.* The $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$ representation $M_{\overline{\mathbb{F}_p}}$ is not admissible, but it admits an increasing exhaustive filtration $F_i M_{\overline{\mathbb{F}_p}}$ by admissible subrepresentations, where F_i consists of those sections with poles of order bounded by i along X^{ss}. To see these are admissible, note that multiplication by H^i gives a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant injection

$$F_i M_{\overline{\mathbb{F}_p}} \hookrightarrow \bigoplus_{k=0}^{p-2} H^0(X, \omega^{k+i(p-1)}). \tag{2.2.1}$$

We deduce that if $\mathbb{T}'$ is a commutative sub-algebra of $\mathbb{T}^{\mathrm{abs}}_{K^p,\overline{\mathbb{F}_p}}$, then $M^{K^p}_{\overline{\mathbb{F}_p}}$ decomposes as a direct sum of generalized eigenspaces

$$M^{K^p}_{\overline{\mathbb{F}_p}} = \bigoplus \left(M^{K^p}_{\overline{\mathbb{F}_p}}\right)_{\mathfrak{m}} \tag{2.2.2}$$

for maximal ideals $\mathfrak{m}$ of $\mathbb{T}'$. Moreover, we find

Lemma 2.2.4. *For $\mathfrak{m}$ a maximal ideal of $\mathbb{T}'$,*

$$\left(M^{K^p}_{\overline{\mathbb{F}_p}}\right)_{\mathfrak{m}} \neq 0 \iff \bigoplus_{k\geq 0} \left(H^0(X_{K^p}, \omega^k)\right)_{\mathfrak{m}} \neq 0.$$

Proof. If $\left(M^{K^p}_{\overline{\mathbb{F}_p}}\right)_{\mathfrak{m}} \neq 0$, then $\left(F_i M^{K^p}_{\overline{\mathbb{F}_p}}\right)_{\mathfrak{m}} \neq 0$ for i sufficiently large, and thus by (2.2.1), $\left(H^0(X, \omega^{k+i(p-1)})\right)_{\mathfrak{m}} \neq 0$. The other direction follows similarly, as $H^0(X_{K^p}, \omega^k)$ injects $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariantly into $M_{\overline{\mathbb{F}_p}}$ by restriction to X_{ord}. □

2.3. **The supersingular Igusa variety.** Let b be the element $\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}$ and let G_0 be the p-divisible group corresponding to the covariant Dieudonné module $\breve{\mathbb{Z}}_p^2$ with Frobenius F acting by $b\sigma$.

The supersingular Igusa variety of Caraiani-Scholze [2], $\mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}}$, represents the functor sending an $\mathbb{F}_p$-algebra R to the set of

$$(E, \varphi_p, \varphi_{\mathbb{A}_f^{(p)}})$$

where $E/\mathrm{Spec} R$ is an elliptic curve up to isogeny, $\varphi_p : E[p^\infty] \to G_0$ is a quasi-isogeny, and $\varphi_{\mathbb{A}_f^{(p)}}$ is a trivialization $V_{\mathbb{A}_f^{(p)}} E \xrightarrow{\sim} \left(\underline{\mathbb{A}_f^{(p)}}\right)^2$.

We write $D_p := \mathrm{End}(\mathbb{X}_b) \otimes \mathbb{Q}_p$, a ramified quaternion algebra over $\mathbb{Q}_p$. There is a natural action of $D_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ on $\mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}}$ by composition with φ_p and $\varphi_{\mathbb{A}_f^{(p)}}$.

2.3.1. *Automorphisms.* To study the Igusa variety, it will be useful to must compute the quasi-isogenies of $E_{0,S}$ and $G_{0,S}$ over an arbitrary base S. We write $\mathrm{QIsog}(E_0)$ (resp. $\mathrm{QIsog}(G_0)$) for the functor sending $S/\overline{\mathbb{F}_p}$ to the group of self quasi-isogenies of $E_0 \times_{\overline{\mathbb{F}_p}} S$ (resp. $G_0 \times_{\overline{\mathbb{F}_p}} S$).

Because $D^\times(\mathbb{Q}) = \mathrm{QIsog}(E_0)(\overline{\mathbb{F}_p})$ and $D_p^\times = \mathrm{QIsog}(G_0)(\overline{\mathbb{F}_p})$, we obtain natural injections

$$\underline{D^\times(\mathbb{Q})} \hookrightarrow \mathrm{QIsog}(E_0),\ \underline{D_p^\times} \hookrightarrow \mathrm{QIsog}(G_0). \tag{2.3.1}$$

Lemma 2.3.2. *The natural injections (2.3.1) are isomorphisms.*

Proof. It suffices to verify this on S-points for $S = \mathrm{Spec} R$. For E_0, we may moreover assume R is of finite type over $\overline{\mathbb{F}_p}$, and then that R is reduced using that quasi-isogenies lift along nilpotent ideals containing p. Then, the result follows from the computation over algebraically closed fields.

For G_0, we cannot argue in the same way as there is no a priori reason for endomorphisms of $G_{0,R}$ to be be defined over a finite type subring (consider, e.g., the endomorphisms of $\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$ over $\mathcal{O}_{\mathbb{C}_p}/p$). Nonetheless, the statement that $\mathrm{QIsog}(G_0) = \underline{D_p}$ is established in a more general context in the proof of [2, Proposition 4.2.11]; the key point is that on $\overline{\mathbb{F}_p}$-algebras, by [2, Lemma 4.1.7, Corollary 4.1.10],

$$R \mapsto \mathrm{End}(G_{0,R})$$

is the Tate module of an étale p-divisible group over $\overline{\mathbb{F}_p}$, and thus equal to the constant sheaf on the $\overline{\mathbb{F}_p}$-points of that Tate module. $\square$

2.3.3. *Punctual uniformization.* We obtain a map

$$\mathrm{unif}_{G_0} : \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}} \to Y \tag{2.3.2}$$

as follows: any point $x \in \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}}(R)$ can be represented by a triple $(E_0, \varphi_p, \varphi_{\mathbb{A}_f^{(p)}})$ such that φ_p induces an *isomorphism* $E_0[p^\infty] \to G_0$, and such a triple is determined up to prime-to-p isogeny of E_0. We map x to the point

$$(E_0, \varphi_{\mathbb{A}_f^{(p)}}) \in Y(R).$$

We write $\mathcal{O}_p := \mathrm{End}(G_0)$, a maximal order in D_p. Arguing as in the proof of Lemma 2.3.2, we find $\underline{\mathcal{O}_p^\times}$ represents the automorphisms of G_0.

Lemma 2.3.4. *The map unif_{G_0} is is a trivializable $\underline{\mathcal{O}_p^\times}$-torsor over Y^{ss}.*

Proof. The map factors through Y^{ss} because the Hasse invariant can be computed in terms of the p-divisible group G_0, and one finds that it is zero. It is also clear from our identification $\mathrm{Aut}(G_0) = \underline{\mathcal{O}_p^\times}$ that the map is a quasi-torsor; to conclude it thus suffices to produce a section.

We write E for the universal elliptic curve up to prime-to-p isogeny over Y^{ss}. Then $E[p^\infty]$ is isomorphic to $G_0 \times Y^{\mathrm{ss}}$: E is pulled back from any finite level, and at finite level Y^{ss} is just a finite union of points. One can choose an isomorphism at each point because each curve parameterized is supersingular and G_0 is the unique height two connected p-divisible group over $\overline{\mathbb{F}_p}$ up-to-isomorphism. The choice of these isomorphisms provides a section, and we conclude. $\square$

2.3.5. *Igusa variety as coset.* We now fix a point

$$x_0 = (E_0, \varphi_{p,0}, \varphi_{\mathbb{A}_f^{(p)},0}) \in \mathrm{Ig}^{\mathrm{ss}}(\overline{\mathbb{F}_p}).$$

We write $D := \mathrm{End}(E_0) \otimes \mathbb{Q}$, a quaternion algebra over $\mathbb{Q}$ ramified at p and ∞. The actions of D on $E_0[p^\infty]$ and $V_{\mathbb{A}_f^{(p)}} E$, transported by $\varphi_{p,0}$ and $\varphi_0^{(p)}$, induce an identification

$$D^\times(\mathbb{A}_f) = D^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^{(p)}) = D_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)}).$$

In particular, we obtain a map $D^\times(\mathbb{Q}) \to D^\times(\mathbb{A}_f)$.

The action on the point x_0 induces an orbit map

$$\underline{D_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})} \to \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}}_p}, \;\; g \mapsto xg. \tag{2.3.3}$$

We will show

Theorem 2.3.6. *The orbit map for the point x_0 factors through a $D^\times(\mathbb{A}_f) = D^\times(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant isomorphism*

$$\underline{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)} = \underline{D^\times(\mathbb{Q})\backslash \left(D_p^\times \times \mathrm{GL}_2\left(\mathbb{A}_f^{(p)}\right)\right)} \xrightarrow{\sim} \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}}_p}.$$

Before proving Theorem 2.3.6, it will be helpful to recall the following basic structural result for quaternionic double cosets. We write $\mathcal{O} = \mathrm{End}(E_0)$, a maximal order in D, and consider the *finite* class set of right fractional ideals in $\mathcal{O}$

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/(\mathcal{O}\otimes\widehat{\mathbb{Z}})^\times = D^\times(\mathbb{Q})\backslash D_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})/\mathcal{O}_p^\times \times \mathrm{GL}_2(\widehat{\mathbb{Z}}^{(p)}).$$

We may fix a finite set of representatives for the class set I and corresponding representatives $\gamma_{\mathcal{I}} \in D^\times(\mathbb{A}_f), \mathcal{I} \in I$ for the double cosets. The stabilizer of $\mathcal{I}$ for left multiplication in $D^\times(\mathbb{Q})$ is the units in a maximal order $\mathcal{O}_{\mathcal{I}}$, and we find

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \cong \bigsqcup_{\mathcal{I}\in I} \mathcal{O}_{\mathcal{I}}^\times \backslash \gamma_{\mathcal{I}}(\mathcal{O}\otimes\widehat{\mathbb{Z}})^\times = \bigsqcup_{\mathcal{I}\in I} \mathcal{O}_{\mathcal{I}}^\times \backslash \mathrm{Isom}(\mathcal{O}\otimes\widehat{\mathbb{Z}}, \mathcal{I}\otimes\widehat{\mathbb{Z}})$$

where the the isomorphisms on the right are of right $\mathcal{O}\otimes\widehat{\mathbb{Z}}$-modules. Because each group $\mathcal{O}_{\mathcal{I}}^\times$ is finite, if we replace $\mathcal{O}\otimes\widehat{\mathbb{Z}}$ with a sufficiently small compact open subgroup $K \subset \mathcal{O}\otimes\widehat{\mathbb{Z}}^\times$, we obtain a finite set of representatives $\gamma_{\mathcal{I},i}$ such that

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \cong \bigsqcup_{\mathcal{I}\in I, i} \gamma_{\mathcal{I},i} K$$

In other words, we obtain a topological splitting of the locally profinite set $D^\times(\mathbb{A}_f)$ as a product of a discrete set and a profinite set,

$$D^\times(\mathbb{A}_f) \cong D^\times(\mathbb{Q}) \times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \tag{2.3.4}$$

compatible with the left action of $D^\times(\mathbb{Q})$ and the right action of K.

Proof of Theorem 2.3.6. By our computation of $\mathrm{QIsog}(E_0)$, we deduce that the orbit map factors as an injection on S points

$$\underline{D^\times(\mathbb{Q})}(S)\backslash \underline{D^\times(\mathbb{A}_f)}(S) \to \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}}_p}(S)$$

From (2.3.4) we deduce

$$\underline{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)}(S) = \underline{D^\times(\mathbb{Q})}(S)\backslash \underline{D^\times(\mathbb{A}_f)}(S),$$

and thus the orbit map factors through an injection

$$\underline{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)} \hookrightarrow \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}}.$$

It remains to show the map is surjective. To do so, it suffices to show that the universal elliptic curve up to isogeny over $\mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}}$ is $E_0 \times \mathrm{Ig}^{\mathrm{ss}}_{\overline{\mathbb{F}_p}}$. But the universal elliptic curve is the pullback from Y^{ss} along unif_{G_0}, and the statement follows as in the proof of Lemma 2.3.4 by reduction to a finite set of points at finite level and the fact that any two supersingular curves over $\overline{\mathbb{F}_p}$ are isogenous. $\square$

Corollary 2.3.7. *The map* unif_{G_0} *factors through an isomorphism*

$$\underline{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/\mathcal{O}_p^\times} \xrightarrow{\sim} Y^{\mathrm{ss}}.$$

Proof. In the decomposition 2.3.4, one can choose K such that $K = \mathcal{O}_p^\times K^p$ for K^p a compact open subgroup in $D^\times(\mathbb{A}_f^{(p)})$. One thus obtains a topological splitting

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \cong \left(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/\mathcal{O}_p^\times\right) \times \mathcal{O}_p^\times$$

compatible with the action of $\mathcal{O}_p^\times$. The result then follows from Lemma 2.3.4. $\square$

Remark 2.3.8. We have avoided the use of any Grothendieck topology because our torsors are all trivializable. In particular, this sidesteps the following question: for X a topological space and Γ a topological group acting on X, for which Grothendieck topologies does $\underline{X}/\underline{\Gamma} = \underline{X/\Gamma}$? Indeed, when $X \to X/\Gamma$ is a trivializable Γ-torsor, this is already true at the level of presheaves.

2.3.9. *Prime-to-p level structure.* If we fix a compact open subgroup K^p, we may consider the variant $\mathrm{Ig}^{\mathrm{ss}}_{K^p,\overline{\mathbb{F}_p}}$ where $\varphi_{\mathbb{A}_f^{(p)}}$ is considered only up to a K^p-orbit. We obtain

Theorem 2.3.10. *The map of Theorem 2.3.6 induces $D_p^\times$-equivariant isomorphisms*

$$\underline{D^\times(\mathbb{Q})\backslash\left(D_p^\times \times \mathrm{GL}_2\left(\mathbb{A}_f^{(p)}\right)\right)/K^p} \xrightarrow{\sim} \mathrm{Ig}^{\mathrm{ss}}_{K^p,\overline{\mathbb{F}_p}}$$

which compile to a $D_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant isomorphism of towers

$$\left(\underline{D^\times(\mathbb{Q})\backslash\left(D_p^\times \times \mathrm{GL}_2\left(\mathbb{A}_f^{(p)}\right)\right)/K^p}\right)_{K^p} \xrightarrow{\sim} \left(\mathrm{Ig}^{\mathrm{ss}}_{K^p,\overline{\mathbb{F}_p}}\right)_{K^p}.$$

Remark 2.3.11. There is no issue here in forming quotients on either side, as there are sections from these finite prime-to-p level versions to the infinite prime-to-p level versions.

2.3.12. *Evaluation of modular forms.* Over Y we have the modular line bundle ω and its powers ω^k. We write $\omega_{G_0} := \mathrm{Lie}(G_0)^*$ with the natural action of $\mathcal{O}_p^\times$. If we pullback to $\mathrm{Ig}^{\mathrm{ss}}$, then we obtain isomorphisms $\omega_{G_0}^k \otimes_{\overline{\mathbb{F}_p}} \mathcal{O}_{\mathrm{Ig}^{\mathrm{ss}}} \xrightarrow{\sim} \mathrm{unif}_{G_0}^* \omega^k$. Moreover, these isomorphism are *equivariant* for the natural actions of $\mathcal{O}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ on both sides. In particular, we obtain a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant isomorphism

$$H^0(Y^{\mathrm{ss}}, \omega^k) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_p^\times}(\mathrm{Lie}(G_0)^k, H^0(\mathrm{Ig}^{\mathrm{ss}}, \mathcal{O})).$$

Note that, by definition, we have an identification of $\mathrm{Lie}G_0$ with $\overline{\mathbb{F}_p}^2/\langle(1,0)\rangle$, and of $\mathcal{O}_p$ with the σ-centralizer of b in $M_2(\breve{\mathbb{Z}}_p)$. In particular, the image of $(0,1)$

gives a basis element of $\mathrm{Lie}G_0$, and $\mathcal{O}_p^\times$ acts on $\mathrm{Lie}G_0 = \overline{\mathbb{F}_p}$ through an explicit character ϵ whose kernel is $\mathcal{N}_p$, the elements congruent to 1 modulo the maximal ideal of $\mathcal{O}_p$. Combining with Theorem 2.3.6, we obtain a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant isomorphism

$$H^0(Y^{\mathrm{ss}}, \omega^k) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_p^\times}\left(\epsilon^k, \mathrm{Cont}\left(D^\times(\mathbb{Q})\backslash D_p \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})/\mathcal{N}_p, \overline{\mathbb{F}_p}\right)\right). \tag{2.3.5}$$

We write

$$\mathcal{A}_{\overline{\mathbb{F}_p}}^{\mathcal{N}_p K^p} := \mathrm{Cont}\left(D^\times(\mathbb{Q})\backslash D_p \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})/\mathcal{N}_p K^p, \overline{\mathbb{F}_p}\right).$$

Then, evaluating homomorphisms in the right-hand side of (2.3.5) on $1 \in \overline{\mathbb{F}_p}$ and passing to K^p-invariants, we obtain Serre's map, a Hecke equivariant isomorphism

$$H^0(Y^{\mathrm{ss}}_{K^p}, \omega^k) \xrightarrow{\sim} \mathcal{A}_{\overline{\mathbb{F}_p}}^{\mathcal{N}_p K^p}[\epsilon^k]. \tag{2.3.6}$$

2.4. **Spectral decomposition of $\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}$.** If K_p is a compact open subgroup of $D^\times(\mathbb{Q}_p)$, then $\mathcal{A}_{\overline{\mathbb{F}_p}}^{K_p}$ is an admissible representation of $\mathrm{GL}_2(\mathbb{A}_f^{(p)}) = D^\times(\mathbb{A}_f^{(p)})$, and these subrepresentations exhaust $\mathcal{A}_{\overline{\mathbb{F}_p}}$ as K_p varies. Thus, as explained in 2.1.3, if $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ is a compact open subgroup and $\mathbb{T}' \subset \mathbb{T}^{\mathrm{abs}}_{K^p} \otimes \overline{\mathbb{F}_p}$ is a commutative sub-algebra, $\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}$ admits a decomposition into generalized eigenspaces

$$\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p} = \bigoplus_{\mathfrak{m}} \left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}\right)_{\mathfrak{m}} \tag{2.4.1}$$

for the action of $\mathbb{T}'$.

Using the results above, we find

Lemma 2.4.1. *For $\mathfrak{m}$ a maximal ideal of $\mathbb{T}'$,*

$$\left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}\right)_{\mathfrak{m}} \neq 0 \iff \left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{\mathcal{N}_p K^p}\right)_{\mathfrak{m}} \neq 0 \iff \bigoplus_{k \geq 0} \left(H^0(X_{K^p}, \omega^k)\right)_{\mathfrak{m}} \neq 0.$$

Proof. Suppose $\bigoplus_{k\geq 0} \left(H^0(X_{K^p}, \omega^k)\right)_{\mathfrak{m}} \neq 0$. For each $k \geq 0$, we have a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant exact sequence

$$0 \to H^0(X, \omega^{k-(p-1)}) \xrightarrow{\cdot H} H^0(X, \omega^k) \xrightarrow{\text{restriction}} H^0(Y^{\mathrm{ss}}, \omega^k).$$

Passing to K^p-invariants and localizing at $\mathfrak{m}$, we obtain

$$0 \to H^0(X_{K^p}, \omega^{k-(p-1)})_{\mathfrak{m}} \xrightarrow{\cdot H} H^0(X_{K^p}, \omega^k)_{\mathfrak{m}} \xrightarrow{\text{restriction}} H^0(Y^{\mathrm{ss}}_{K^p}, \omega^k)_{\mathfrak{m}}. \tag{2.4.2}$$

By induction on k, we deduce that if $H^0(X, \omega^k)_{\mathfrak{m}} \neq 0$ then $H^0(Y^{\mathrm{ss}}_{K^p}, \omega^{k'})_{\mathfrak{m}} \neq 0$ for some $0 \leq k' \leq k$, and, applying Serre's isomorphism (2.3.6), that $\left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}\right)_{\mathfrak{m}} \neq 0$.

Suppose $\left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}\right)_{\mathfrak{m}} \neq 0$. Then, because $\mathcal{N}_p$ is a pro-p group and $\left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}\right)_{\mathfrak{m}} \subset \mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}$ is an admissible characteristic p representation of $D_p^\times$, it admits a nonzero $\mathcal{N}_p$-fixed vector, and thus $\left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{\mathcal{N}_p K^p}\right)_{\mathfrak{m}} \neq 0$.

Finally, suppose $\left(\mathcal{A}_{\overline{\mathbb{F}_p}}^{K^p}\right)_{\mathfrak{m}} \neq 0$. Then, as

$$\mathcal{A}_{\overline{\mathbb{F}_p}}^{\mathcal{N}_p K^p} = \bigoplus_{k \in \mathbb{Z}/(p^2-1)\mathbb{Z}} \mathcal{A}_{\overline{\mathbb{F}_p}}^{\mathcal{N}_p K^p}[\epsilon^k],$$

we deduce that for some some $k \geq 0$, $\mathcal{A}_{\overline{\mathbb{F}}_p}^{\mathcal{N}_p K^p}[\epsilon^k]_{\mathfrak{m}} \neq 0$. Because ω is ample on X_{K^p} and ϵ^k only depends on k mod p^2-1, we may choose this value of k large enough that $H^1(X_{K^p}, \omega^k) = 0$. Then, the sequence (2.4.2) extends to a short exact sequence; in particular, restriction induces a surjection

$$H^0(X_{K^p}, \omega^k)_{\mathfrak{m}} \twoheadrightarrow H^0(Y^{\mathrm{ss}}_{K^p}, \omega^k)_{\mathfrak{m}}.$$

Applying Serre's isomorphism (2.3.6) we obtain $H^0(Y^{\mathrm{ss}}_{K^p}, \omega^k)_{\mathfrak{m}} \neq 0$, and thus $H^0(X_{K^p}, \omega^k)_{\mathfrak{m}} \neq 0$.

□

Remark 2.4.2. If we compare Lemma 2.4.1 and Lemma 2.2.4, then the missing third space in the statement of Lemma 2.2.4 is the space of functions on the (partially compactified, big, ordinary) Igusa variety of Caraiani-Scholze [2, 3], which parameterizes isomorphisms $E[p^\infty] \to \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$. One can again use a group theoretic argument to show that the Hecke eigensystems here are the same as those appearing in $\mathrm{Ig}^{\mathrm{ord}}_{1,K^p}$ – in classical terms, one uses Frobenius to reduce to the small Igusa variety of Katz, then reduces to $\mathrm{Ig}^{\mathrm{ord}}_{1,K^p}$ by taking invariants under the pro-p group $1+p\mathbb{Z}_p$.

2.4.3. *Consequences.* Combining Lemmas 2.2.4 and 2.4.1, we obtain Theorem 1.1.1 of the introduction; only a finite number of maximal ideals can appear in the decompositions because $\mathcal{A}^{\mathcal{N}_p K^p}$ is a finite dimensional $\overline{\mathbb{F}_p}$-vector space. We note that this also proves a theorem of Jochnowitz [16] stating that there are only finitely many eigensystems appearing in mod p modular forms.

3. The spectral p-adic Jacquet-Langlands correspondence

In this section we prove Theorem A: After recalling some preliminaries on perfectoid modular curve, modular forms, and p-adic modular forms in 3.1 and 3.2, in 3.3 we recall the relation between the perfectoid Igusa variety and the Hodge-Tate period map. In 3.4 we use this relation to construct an evaluation map from classical modular forms to $\mathcal{A}_{\mathbb{C}_p}$. The key properties of this evaluation map are that it is injective and has dense image; these are established in Theorem 3.4.1. To finish the proof, we must then take a brief detour in 3.5 to establish some basic results on completed Hecke algebras. We conclude in Theorem 3.6.1 by giving the isomorphism of completed Hecke algebras of Theorem A along with some other well-known isomorphisms.

3.1. **Perfectoid modular curves.** For $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ a sufficiently small compact open subgroup, we write Y_K for the modular curve over $\mathbb{C}_p$ parameterizing elliptic curves up to isogeny equipped with K-level structure, i.e. a K-orbit of trivializations $\underline{\mathbb{A}_f^2} \to V_f E$ (cf. [4] for a comparison of the up-to-isogeny and up-to-isomorphism moduli problems). We write X_K for its compactification.

We view both Y and X as adic spaces over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$, and we consider the infinite level perfectoid modular curves as in [21]

$$Y \sim \lim Y_K, X \sim \lim X_K.$$

In loc. cit. one takes the limit with fixed level away from p, but there is no harm in further taking the limit over prime-to-p level. The action of $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$ on the tower of Y_K by change of level (resp. its unique extension to the tower of X_K) induces an action on Y and X.

3.1.1. *Modular forms.* For any K as above there is a universal elliptic curve up to isogeny $\pi : E \to Y_K$. We consider the modular sheaf $\omega := \pi_* \Omega_{E/Y_K}$, and its extension to X given by taking sections with holomorphic q-expansions at the cusps, which we denote also by ω. These assemble to a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant family of line bundles on the tower of X_K.

Pulling back ω to infinite level gives a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant line bundle on X, which we also denote ω. For K a sufficiently small compact open as above, we have $H^0(X, \omega^k)^K = H^0(X_K, \omega^k)$ (the classical space of modular forms over $\mathbb{C}_p$) – indeed, it suffices to verify a similar identity locally in the analytic topology on Y_K, where it follows from the sheaf property of the completed structure sheaf on the pro-étale site of Y_K.

We write $M_{k,\mathbb{C}_p} := \mathrm{colim}_K H^0(X_K, \omega^k)$, an admissible representation of $\mathrm{GL}_2(\mathbb{A}_f)$. By the above discussion, pullback to X identifies $M_{k,\mathbb{C}_p}$ with $H^0(X, \omega^k)^{\mathrm{sm}}$, the set of smooth vectors for the $\mathrm{GL}_2(\mathbb{A}_f)$-action on $H^0(X, \omega^k)$.

Remark 3.1.2. As in the mod p case, to obtain a $\mathrm{GL}_2(\mathbb{A}_f)$-action inducing the standard Hecke action, one should replace ω^k with $\Omega(\text{cusps}) \otimes \omega^{k-2}$. Again, this because the Kodaira spencer isomorphism

$$\omega^2 = \Omega(\text{cusps})$$

introduces a twist by the unramified determinant character. This twisting factor is trivial on any compact open and serves only to replace each Hecke operator with a multiple. In particular, after passing to $\mathbb{Z}_p$, the multiples appearing in the prime-to-p Hecke action become invertible, and thus will not change the completed Hecke algebra. In particular, it is reasonable to include the twist only "when convenient," that is, when comparing with singular/étale cohomology.

3.1.3. *The Hodge-Tate period map.* In [21], Scholze constructs the Hodge-Tate period map,

$$\pi_{\mathrm{HT}} : X \to \mathbb{P}^1_{\mathbb{C}_p}.$$

Over Y, it is the classifying map for the line

$$\mathrm{Lie}E(1) \subset V_p E \otimes \widehat{\mathcal{O}} \cong \mathcal{O}^2$$

given by the Hodge-Tate filtration for the universal elliptic curve and the trivialization of its relative Tate module.

The map π_{HT} is $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant, and there is a natural isomorphism of $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant line bundles

$$\pi_{\mathrm{HT}}^* \mathcal{O}(1) = \omega \otimes \mathbb{Q}_p(-1).$$

We fix a trivialization of $\mathbb{Q}_p(-1)$ by choosing a compatible system of roots of unity in $\mathbb{C}_p$ in order to identify

$$\pi_{\mathrm{HT}}^* \mathcal{O}(1) = \omega$$

as $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant bundles.

Remark 3.1.4. $\mathbb{Q}_p(-1)$ can also be trivialized on the perfectoid modular curve at infinite level over $\mathbb{Q}_p$ (rather than $\mathbb{C}_p$) by using the Weil pairing. However, this trivialization is not $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant – instead it introduces a twist.

3.2. **p-adic modular forms.** If we fix a set of cusps $c_1, \ldots, c_m$, one in each connected component of $X_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \breve{\mathbb{Q}}_p}$, then we obtain q-expansion injections

$$M_{k,\breve{\mathbb{Q}}_p}^{\mathrm{GL}_2(\mathbb{Z}_p)K^p} \to \prod_{i=1}^{m} \breve{\mathbb{Z}}_p[[q]][1/p]$$

Following Serre [26], we define the space of p-adic modular forms of level K^p, $\mathbb{V}_{\breve{\mathbb{Q}}_p}^{K^p}$, to be the $\breve{\mathbb{Q}}_p$-Banach space given by completing the joint image of these q-expansion maps over all k for the natural Banach topology on $\breve{\mathbb{Z}}_p[[q]][1/p]$. The prime-to-$p$ abstract Hecke algebra $\mathbb{T}_{\mathrm{abs}}$ acts uniformly on q-expansions in this topology, and thus the action extends to $\mathbb{V}_{\breve{\mathbb{Q}}_p}^{K^p}$. There is a surjective Hecke equivariant reduction map from the unit ball in p-adic modular forms to mod p modular forms.

Remark 3.2.1. Just as mod p modular forms can be realized as functions on the first level of the ordinary Igusa tower, Katz has shown that p-adic modular forms can be constructed as the space of functions that are holomorphic at the cusps on the Katz-Igusa formal scheme parameterizing isomorphisms $\widehat{E} \to \widehat{\mathbb{G}_m}$. Indeed, this is essentially [18, Theorem 2.1], except that Katz works in the setting of connected modular curves at full prime-to-p level $\Gamma(N)$.

3.3. **The perfectoid supersingular Igusa variety.** Recall that in 2.3 we fixed a model connected height two p-divisible group G_0 over $\overline{\mathbb{F}_p}$. The perfectoid supersingular Igusa variety $\mathrm{Ig}_{\mathbb{C}_p}^{\mathrm{ss}}$ as in [2, Section 4] is the perfectoid space over $\mathbb{C}_p$ whose points on an affinoid perfectoid (R, R^+) over $\mathbb{C}_p$ are identified with triples

$$(E, \varphi_p, \varphi^{(p)})$$

where $E/(R^+/p)$ is an elliptic curve up to quasi-isogeny,

$$\varphi_p : E[p^\infty] \to G_0$$

is a quasi-isogeny, and $\varphi^{(p)}$ is a trivialization of $V_{\mathbb{A}_f^{(p)}} E$.

It is the adic generic fiber (in the sense of [23]) of the base change to $\mathcal{O}_{\mathbb{C}_p}$ of the Witt lift of $\mathrm{Ig}_{\overline{\mathbb{F}_p}}^{\mathrm{ss}}$ (a p-adic formal scheme). By Theorem 2.3.6, we deduce that it is equivariantly isomorphic to the profinite set

$$D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)$$

viewed as a perfectoid space; in other words,

$$\mathrm{Ig}_{\mathbb{C}_p}^{\mathrm{ss}} \cong \mathrm{Spa}(\mathcal{A}_{\mathbb{C}_p}, \mathcal{A}_{\mathcal{O}_{\mathbb{C}_p}}).$$

3.3.1. *Uniformization of fibers of* π_{HT}. We fix now a p-divisible group $G/\mathcal{O}_{\mathbb{C}_p}$, a quasi-isogeny

$$\rho_G : G_{\mathcal{O}_{\mathbb{C}_p}} \xrightarrow{\sim} G_{0, \mathcal{O}_{\mathbb{C}_p}/p},$$

and a trivialization $\varphi_{p,\eta} : T_p G(\mathcal{O}_{\mathbb{C}_p}) \xrightarrow{\sim} \mathbb{Z}_p^2$. By the Scholze-Weinstein classification [23], the pair $(G, \varphi_{p,\eta})$ is equivalent to a point $x \in \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(\mathbb{Q}_p)$. As it will be convenient later, we chose our data so that x is in the affinoid ball $B_1 : |X/Y| \le 1$. The lifting data ρ_G determines a point $x_\infty \in \widehat{\mathcal{M}}_{\mathrm{LT},\infty}(\mathbb{C}_p)$ of the perfectoid Lubin-Tate space lying in $\pi_{\mathrm{HT,loc}}^{-1}(x)$.

From this data we obtain a map

$$\mathrm{unif}_{x_\infty} : \mathrm{Ig}_{\mathbb{C}_p}^{\mathrm{ss}} \to X$$

On points in an affinoid perfectoid (R, R^+), it sends $(E, \varphi_p, \varphi^{(p)})$ to the generic fiber of the lift of E to R^+ determined through Serre-Tate lifting theory (cf. [17, Section 1]) by

$$\rho_G^{-1} \circ \varphi_p : E[p^\infty] \to G_{R^+/p},$$

equipped with the level structure determined by $\varphi_{p,\eta}$ and the unique lift of $\varphi^{(p)}$.

By construction, unif_{x_∞} factors through the closed subset $\pi_{\mathrm{HT}}^{-1}(x) \subset X_{\infty,\mathbb{C}_p}$. The latter is a Zariski closed subset of a perfectoid space, and thus admits a canonical structure of a perfectoid space through which unif_{x_∞} factors (because the domain is also a perfectoid space).

In addition to the obvious $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariance, the map unif_{x_∞} also satisfies an equivariance at p: if we write T_x for the group of non-zero quasi-isogenies of G, then $\varphi_{p,\eta}$ identifies T_x with the stabilizer of x in $\mathrm{GL}_2(\mathbb{Q}_p)$, and ρ_G identifes T_x with a subgroup of $D_p^\times$. In particular, T_x acts on $\pi_{\mathrm{HT}}^{-1}(x)$ as well as $\mathrm{Ig}^{\mathrm{ss}}_{\mathbb{C}_p}$, and unif_{x_∞} is equivariant for these actions. The group T_x is equal to either $\mathbb{Q}_p^\times$ or $F^\times$ for a quadratic extension $F/\mathbb{Q}_p$; the latter occurs exactly when $x \in \mathbb{P}^1(F) - \mathbb{P}^1(\mathbb{Q}_p)$.

Theorem 3.3.2. *The map* unif_{x_∞} *induces a* $T_x \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$*-equivariant isomorphism of perfectoid spaces*

$$\mathrm{unif}_{x_\infty} : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \cong \mathrm{Ig}^{\mathrm{ss}}_{\mathbb{C}_p} \xrightarrow{\sim} \pi_{\mathrm{HT}}^{-1}(x).$$

Proof. The map unif_{x_∞} is the restriction to $\mathrm{Ig}^{\mathrm{ss}}_{\mathbb{C}_p} \times x_\infty$ of a map

$$\mathrm{unif} : \mathrm{Ig}^{\mathrm{ss}}_{\mathbb{C}_p} \times_{\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})} \widehat{\mathcal{M}}_{\mathrm{LT},\infty,\mathbb{C}_p} \to X$$

defined similarly. There is a local Hodge-Tate period map

$$\pi_{\mathrm{HT,loc}} : \widehat{\mathcal{M}}_{\mathrm{LT},\infty,\mathbb{C}_p} \to \mathbb{P}^1_{\mathbb{C}_p},$$

and by [2, Lemma 4.3.20] (cf. also [2, Definition 4.3.17]), the diagram

$$\begin{tikzcd}
\mathrm{Ig}_{K^p,\mathbb{C}_p} \times_{\mathrm{Spa}(\mathbb{C}_p,\mathcal{O}_{\mathbb{C}_p})} \widehat{\mathcal{M}}_{\mathrm{LT},\infty,\mathbb{C}_p} \arrow[r] \arrow[d, "\mathrm{unif}"] & \widehat{\mathcal{M}}_{\mathrm{LT},\infty,\mathbb{C}_p} \arrow[d, "\pi_{\mathrm{HT,loc}}"] \\
X_{\infty K^p,\mathbb{C}_p} \arrow[r, "\pi_{\mathrm{HT}}"] & \mathbb{P}^1
\end{tikzcd}$$

is Cartesian on perfectoid spaces. Thus, because $\pi_{\mathrm{HT,loc}}(x_\infty) = x$, we find that unif_{x_∞} induces an isomorphism between the functors of points of $\mathrm{Ig}_{K^p,\mathbb{C}_p}$ and $\pi_{\mathrm{HT}}^{-1}(x)$ on perfectoid spaces. Because both are perfectoid spaces, and a perfectoid space is determined by its functor of points on perfectoid spaces, we conclude unif_{x_∞} is an isomorphism. □

3.4. **Evaluation of modular forms.** Any basis for the fiber of $\mathcal{O}(1)$ at $x \in \mathbb{P}^1(\mathbb{C}_p)$ induces a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant trivialization of $\omega|_{\pi_{\mathrm{HT}}^{-1}(x)}$. In particular, if we take as basis the section $Y|_x$, we obtain a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant map from classical modular forms

$$\mathrm{eval}_{x_\infty} : \bigoplus_{k \geq 0} M_{k,\mathbb{C}_p} \to \mathcal{A}_{\mathbb{C}_p} \left(= H^0(\mathrm{Ig}_{\mathbb{C}_p}, \mathcal{O}) \right).$$

Theorem 3.4.1. *The map* eval_{x_∞} *is injective and has dense image. Moreover, for any compact open* $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ *the restriction*

$$\mathrm{eval}_{x_\infty}^{K^p} : \bigoplus_{k\geq 0} M_{k,\mathbb{C}_p}^{K^p} \to \mathcal{A}_{\mathbb{C}_p}^{K^p}$$

is also injective with dense image.

Proof. We first show injectivity: because $z \in \mathbb{Z}_p^\times \subset T_x$ acts as multiplication by z^k on our fixed trivialization $Y|_x$, we deduce that for any compact open $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$, the image of $M_{k,\mathbb{C}_p}^{K_p}$ transforms under z^k for the action of $z \in K_p \cap \mathbb{Z}_p^\times$. Thus, the image of $M_{k,\mathbb{C}_p}$ lands in the subspace $\mathcal{A}_{\mathbb{C}_p}[k]$ of vectors on which the action of the central $\mathbb{Z}_p^\times$ is differentiable with derivative k. In particular, there can be no cancellation between the different degrees k.

To show the map is injective on each $M_{k,\mathbb{C}_p}$, we first observe that for $K = K_pK^p$ a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, the image of $\pi_{\mathrm{HT}}^{-1}(x)$ in $X_K(\mathbb{C}_p)$ intersects every connected component of $X_{K,\mathbb{C}_p}$ in an infinite set – indeed, the map factors as an injection from $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/(K_p \cap T_x)\cdot K^p$, and the connected component of the image in $X_{K,\mathbb{C}_p}$ is recorded by the map

$$g = g_p \times (g_\ell)_{\ell\neq p} \mapsto \mathrm{Nrd}\, g_p \cdot |\,\mathrm{Nrd}\, g_p|_p \cdot \prod_{\ell \neq p} |\mathrm{det} g_\ell|_\ell$$

with values in $\mathbb{Z}_p^\times$ modulo the image of K.

If s is a non-zero section of a line bundle on $X_{K,\mathbb{C}_p}$ then there is at least one connected component where it has only finitely many zeroes. Thus, any section of ω^k over X_K which vanishes upon restriction to $\pi_{\mathrm{HT}}^{-1}(x)$ is identically zero, and we conclude the map is injective.

We now show the map has dense image. By assumption, x is contained in the affinoid

$$B_1 : |X|/|Y| \leq 1 \subset \mathbb{P}^1,$$

and the results of [21] give that $U := \pi_{\mathrm{HT}}^{-1}(B_1)$ is affinoid perfectoid. By [1, Remark 7.5] (Zariski closed implies strongly Zariski closed; cf. also [21, Definition II.2.6]), we find that the map

$$H^0(U, \mathcal{O}^+) \xrightarrow{\sim} H^0(\pi_{\mathrm{HT}}^{-1}(x), \mathcal{O}^+) = \mathcal{A}_{\mathcal{O}_{\mathbb{C}_p}}$$

is almost surjective; in particular, the image contains $p\mathcal{A}_{\mathcal{O}_{\mathbb{C}_p}}$.

Thus, given $f \in p\cdot \mathcal{A}_{\mathcal{O}_{\mathbb{C}_p}}$, we can lift it to $\tilde{f} \in H^0(U, \mathcal{O}^+)$. By the method of fake Hasse invariants (cf. [21, Proof of Theorem 4.3.1, pp. 1028-1031]), we find that for any $n > 0$ there is a k sufficiently large and K sufficiently small such that $Y^k\tilde{f}$ is approximated mod p^n on U by an element of $\alpha \in M_{k,\mathbb{C}_p}^K$: concretely, this means

$$\alpha/Y^k \in H^0(U, \mathcal{O}^+)$$

and is equal to $\tilde{f}$ mod p^n. Thus, the image of α under eval_{x_∞} is equal to f mod p^n, and we conclude the image is dense. The same argument can be run using everywhere the perfectoid space with fixed prime-to-p level K^p, giving the result also on K^p-invariants.

□

3.5. **Completed Hecke algebras.** In order to deduce Theorem A from Theorem 3.4.1, we must first take a detour to develop some tools for comparing completed Hecke algebras. To clarify the statements, we abstract to the setting of general algebra actions on p-adic Banach spaces; we refer the reader to the introduction of [25] for basic definitions and results on Banach spaces over non-archimedean fields.

3.5.1. *Completing actions.*

Definition 3.5.2. An action of a (not necessarily commutative) ring A by bounded operators on a Banach space V is *uniform* if for all $a \in A$ and $v \in V$,

$$||a \cdot v|| \leq ||v||.$$

Definition 3.5.3. If A is a ring, K is a non-archimedean field, and (W_i) is family of Banach spaces equipped with uniform actions of A, the *completion*[2] of A with respect to $(W_i)_{i \in I}$ is the closure $\widehat{A}$ of the image of A in

$$\prod_{i \in I} \mathrm{End}_{\mathrm{cont}}(W_i)$$

where each $\mathrm{End}_{\mathrm{cont}}(W_i)$ is equipped with the strong operator topology (the topology of pointwise convergence for the Banach topology on W_i) and the product is equipped with the product topology.

Remark 3.5.4. We highlight that the definition of $\widehat{A}$ does not depend on the specific norm on each W_i, only on the underlying topology.

We give two equivalent characterizations of the elements of $\widehat{A}$:

Lemma 3.5.5. *In the setting of Definition 3.5.3:*

(Nets): $\prod_i f_i \subset \widehat{A}$ *if and only if there exists a net* $a_j \in A$ *such that for any* $i \in I$ *and any* $w \in W_i$,

$$\lim a_j \cdot w = f_i(w).$$

(Congruences): *For each* $i \in I$, *fix a choice* W_i° *of a lattice in* W_i *preserved by* A *(e.g., the elements of norm* ≤ 1*). Then,* $\prod_i f_i \in \widehat{A}$ *if and only if* f_i *preserves* W_i° *for each* i*, and for any finite subset* $S \subset I$ *and any topologically nilpotent* $\pi \in K$*, there exists* $a \in A$ *such that for each* $i \in S$*,* a *and* f_i *have the same image in*

$$\mathrm{End}(W_i^\circ/\pi).$$

Proof. The characterization (Nets) is immediate from the definition of the strong operator topology as the topology of pointwise convergence of nets and the characterization of the product topology as the topology of term-wise convergence of nets.

The characterization (Congruences) then follows by considering nets on the directed set of finite subsets of I times $\mathbb{N}$ (where $\mathbb{N}$ is interpreted as the power of some fixed uniformizer) to show that (Congruences) implies (Nets). $\square$

[2] in the literature on Hecke algebras this is sometimes referred to as the weak completion; we avoid this terminology because of a conflict with terminology in functional analysis, where this is the completion for the strong operator topology.

Using either the characterization in terms of nets plus uniformity of the action, or the characterization in terms of congruences, we find that $\widehat{A}$ is again a ring. It is equipped with a natural structure as an A-algebra.

Remark 3.5.6. By (Congruences), we can also construct $\widehat{A}$ as the closure of the image of A in

$$\prod \mathrm{End}(W_i^\circ/\pi)$$

where the product is over all possible choices of $i \in I$, a lattice $W_i^\circ \subset W_i$, and a topologically nilpotent π, and each term has the discrete topology.

3.5.7. *Relating completions.* The following lemma says that completion is insensitive to base extension. This is useful for us as our comparisons of Hecke-modules take place over large extensions of $\mathbb{Q}_p$, whereas one is typically interested in Hecke algebras over $\mathbb{Z}_p$.

Lemma 3.5.8. *Let $K \subset K'$ be an extension of complete non-archimedean fields, and let A be a (not-necessarily commutative) ring. Suppose (W_i) is a family of orthonormalizable Banach spaces over K equipped with uniform actions of A. Then the identity map $A \to A$ extends to a topological isomorphism between the completion of A acting on (W_i) and the completion of A acting on $(W_i \widehat{\otimes}_K K')$.*

Proof. We note that for a bounded net ϕ_j of bounded operators on a orthonormalizable Banach space, $\phi_j \to f$ in the strong operator topology if and only if $\phi_j(e) \to f(e)$ for any element e of a fixed orthonormal basis.

In particular, because an orthonormal basis for W_i is also an orthonormal basis for $W_i \otimes_K K'$, we find that the completion for (W_i) injects into the completion for $(W_i \widehat{\otimes}_K K')$. More over, since W_i is closed inside of $W_i \widehat{\otimes}_K K'$ and preserved by A, we find that for any net $a_j \in \mathbb{T}'$ and element e in the orthonormal basis, $\lim_j a_j(e)$ is in W_i if it exists. Thus, an element in the completion for $(W_i \widehat{\otimes}_K K')$ comes from an element in the completion for (W_i). □

The following lemma is our main technical tool for comparing completed Hecke algebras. It says that the completion is determined by any family of invariant subspaces whose sum is dense.

Lemma 3.5.9. *Let K be a non-archimedean field, and let A be a (not-necessarily commutative) ring. Suppose V is an orthonormalizable K-Banach space equipped with a uniform action of A, and $(W_i)_{i \in I}$ is a family of topological vector spaces over K equipped with A-actions and continuous A-equivariant topological immersions*

$$\psi_i : W_i \hookrightarrow V.$$

If $\sum \mathrm{Im}\psi_i$ is dense in V, then the identity map on A induces an isomorphism between the weak completion of A acting on $(W_i)_{i \in I}$ and the weak completion of A acting on V.

Remark 3.5.10. In this setup, the action of A on W_i is automatically uniform for the restriction to W_i of the norm on V, which, by hypothesis, induces the same topology.

Proof. Denote by $\widehat{A}_V \subset \mathrm{End}(V)$ the strong completion of A acting on V, and $\widehat{A}_W \subset \prod \mathrm{End}(W_i)$ the strong completion of A acting on $(W_i)_{i \in I}$.

We first show there is a map $\widehat{A}_V \to \widehat{A}_W$ extending the identity map $A \to A$: Let $\phi \in \widehat{A}_V$, and let ϕ_j be a net in the image of A approaching ϕ. For $w \in W_i$ (considered as closed subspace of V via ψ_i),

$$\phi(w) = \lim_j \phi_j(w).$$

For each j, $\phi_j(w)$ is contained in W_i by the A-equivariance of ψ_i, and thus, since W_i is closed, $\phi(w) \in W_i$. Thus, ϕ preserves W_i. Using this, we obtain a map

$$\widehat{A}_V \to \prod_i \mathrm{End}(W_i)$$

extending the map $A \to \prod_i \mathrm{End}(W_i)$. Furthermore, it follows immediately that the image lies in $\widehat{A}_W$.

The map is injective by the density of $\sum W_i \subset V$. We show now that it is surjective. By the density of $\sum \mathrm{Im}\psi_i$, we may choose an orthonormal basis for V contained in the image of $\oplus W_i$. A bounded net of operators in $\mathrm{End}(V)$ converges if and only if it converges on each element of an orthonormal basis. Now, if $\phi \in \widehat{A}_W$ is the limit of a net ϕ_j in the image of A, then we see that $\phi_j(e)$ converges for each element e of the orthonormal basis, and thus ϕ_j also converges in $\mathrm{End}(V)$, and its limit maps to ϕ, as desired.

Thus the map $\widehat{A}_V \to \widehat{A}_W$ is bijective. By similar arguments, the weak topologies agree, and thus the map is a topological isomorphism. □

3.6. **Comparison of completed Hecke algebras.** We now prove Theorem A: Let $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^{(p)}) = D^\times(\mathbb{A}_f^{(p)})$ be a compact open subset, let

$$\mathbb{T}^{\mathrm{abs}}_{K^p} = \mathbb{Z}_p[K^p \backslash GL_2(\mathbb{A}_f^{(p)})/K^p],$$

and let $\mathbb{T}' \subset \mathbb{T}^{\mathrm{abs}}$ be a commutative $\mathbb{Z}_p$-subalgebra. Combining Theorem 3.4.1 and Lemma 3.5.9, we conclude that the completion of $\mathbb{T}'$ acting on $(M_{k,\mathbb{C}_p}^{K_p K^p})_{K_p,k}$ is equal to the completion of $\mathbb{T}'$ acting on $\mathcal{A}_{\mathbb{C}_p}^{K^p}$. Note that all of these are naturally the base extensions of Hecke modules defined over $\mathbb{Q}_p$, and invoking Lemma 3.5.8, we find that we can replace $\mathbb{C}_p$ with $\mathbb{Q}_p$ on both sides (to apply the result, note that $\mathcal{A}_{\mathbb{Q}_p}^{K^p}$ admits an orthonormal basis by lifting any algebraic basis of $\mathcal{A}_{\mathbb{F}_p}^{K^p}$).

On the other hand, combining the definition of the space of p-adic modular forms $\mathbb{V}_{\breve{\mathbb{Q}}_p}^{K^p}$ (cf. 3.2) and Lemma 3.5.9, we find that the completion of $\mathbb{T}'$ acting on $\mathbb{V}_{\breve{\mathbb{Q}}_p}^{K^p}$ is equal to the completion of $\mathbb{T}'$ acting on $(M_{k,\breve{\mathbb{Q}}_p}^{\mathrm{GL}_2(\mathbb{Z}_p)K^p})_k$, and again by Lemma 3.5.8 we may replace $\breve{\mathbb{Q}}_p$ with $\mathbb{Q}_p$ in the latter.

Thus, to deduce Theorem A, that is, that the completed Hecke algebra for the action of $\mathbb{T}'$ on $\mathcal{A}_{\mathbb{Q}_p}^{K^p}$ is equal to the completed Hecke algebra for the action of $\mathbb{T}'$ on $\mathbb{V}_{\breve{\mathbb{Q}}_p}^{K^p}$, we must show that the completed Hecke algebra for modular forms of all weights k and fixed p-level $\mathrm{GL}_2(\mathbb{Z}_p)$ is equal to the completed Hecke algebra for modular forms of all weights k and varying p-level. This is basically a well-known result of Hida [10, Equation (1.7)], however, we are not aware of a full proof in the literature, or even a statement written down in suitable generality for the present application. Thus we include a proof here, arguing with completed cohomology as explained by Emerton [6, Remarks 5.4.2 and 5.4.3].

Theorem 3.6.1. *The identity map $\mathbb{T}' \to \mathbb{T}'$ extends to a homeomorphism of the completed Hecke algebras $\mathbb{T}'$ acting on the following:*

(1) $\left(M_k^{K_pK^p}\right)_{k,K_p}$ *for k varying over all non-negative integers and K_p varying over all compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$,*
(2) $\left(M_k^{\mathrm{GL}_2(\mathbb{Z}_p)K^p}\right)_k$ *for k varying over all non-negative integers, and*
(3) $\left(M_2^{K_pK^p}\right)$*, for K_p varying over all compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.*
(4) *The completed cohomology of the modular curve at level K^p (cf. [5]).*

$$\widehat{H}^1_{K^p} := \left(\varprojlim_m \varinjlim_{K_p} H^i(Y_{K_pK^p}(\mathbb{C}), \mathbb{Z}/p^m\mathbb{Z})\right)[1/p]$$

(5) *The space of quaternionic automorphic forms $\mathcal{A}^{K^p}_{\mathbb{Q}_p}$*
(6) *The space of p-adic modular forms $\mathbb{V}^{K^p}_{\breve{\mathbb{Q}}_p}$.*

Proof. We have already established the identities of completed Hecke algebras (1)=(5) and (2)=(6). We will conclude by identifying (4)=(1), (4)=(2), and (4)=(3).

Let $\mathrm{pr} : E_K \to Y_K(\mathbb{C})$ be the univeral elliptic curve over $Y_K(\mathbb{C})$ and let $\underline{\mathrm{Sym}^k}$ be the kth symmetric power of $R^1\mathrm{pr}_*\mathbb{Z}$, a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant local system on the tower $(Y_K(\mathbb{C}))_K$. From our fixed isomorphism $\overline{\mathbb{Q}_p} \cong \mathbb{C}$, we obtain for each $k \geq 2$ and K_p a $\mathbb{T}^{\mathrm{abs}}$-equivariant injection

$$M^{K_pK^p}_{k,\mathbb{Q}_p} \hookrightarrow H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{C}_p \tag{3.6.1}$$

given by composing the maps

$$\begin{aligned} M^{K_pK^p}_{k,\mathbb{Q}_p} \hookrightarrow M^{K_pK^p}_{k,\mathbb{C}} \hookrightarrow H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{C} \\ \hookrightarrow H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{C}_p, \end{aligned}$$

where the second arrow comes from the classical Eichler-Shimura isomorphism and the last arrow comes from composition of the isomorphism $\overline{\mathbb{Q}_p} \cong \mathbb{C}$ with $\overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}_p$. It follows from the Eichler-Shimura isomorphism that (3.6.1) induces an isomorphism on the image of $\mathbb{T}'$ in the respective endomorphism rings. If we denote $\widehat{\mathbb{T}'_{\mathrm{aux}}}$ the completed Hecke algebra for $\mathbb{T}'$ acting on the family

$$\left(H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{C}_p\right)_{K_p,k},$$

then we deduce that $\widehat{\mathbb{T}'_{\mathrm{aux}}}$ is isomorphic to the completed Hecke algebra for (1). By Lemma 3.5.8, $\widehat{\mathbb{T}'_{\mathrm{aux}}}$ is also the completed Hecke algebra for $\mathbb{T}'$ acting on

$$\left(H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{Q}_p\right)_{K_p,k}.$$

As in [5, 6][3]

$$H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Hom}_{K_p}(\mathrm{Sym}^{k-2}\mathbb{Q}_p^2, \widehat{H^1}_{K^p}).$$

[3]Note that our normalizations for actions are different, so that we obtain a Sym^{k-2} in the source of the Hom whereas in [5, 6] there is a $(\mathrm{Sym}^{k-2})^*$ (cf. [6, last paragraph of Section 2].)

Thus, if we fix for each $k \geq 2$ a non-zero vector in $(\mathrm{Sym}^{k-2}\mathbb{Q}_p^2)$, then pairing with these vectors gives Hecke-equivariant injections

$$H^1(Y_{K_pK^p}(\mathbb{C}), \underline{\mathrm{Sym}^{k-2}}) \otimes \mathbb{Q}_p \hookrightarrow \widehat{H}^1_{K^p}$$

whose joint image is dense (indeed, it is already so if we fix $k = 2$), and thus we deduce from Lemma 3.5.9 that $\widehat{\mathbb{T}'_{\mathrm{aux}}}$ is isomorphic to the completed Hecke algebra for (4), establishing (1)=(4). Then, running the same argument using only weight two modular forms, we find (2)=(4).

Arguing similarly and using the density of $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors in $\widehat{H}^1_{K^p}$ (specifically of the ones which transform locally as $\mathrm{Sym}^{k-2}\mathbb{Q}_p^2$ for some k; we do not need to also allow for arbitrary twists by a determinant), we obtain (3)=(4). □

Remark 3.6.2. From our perspective (cf. Remark 2.4.2), instead of p-adic modular forms it is more natural to consider the larger space of p-adic automorphic forms given by functions on the Caraiani-Scholze Igusa formal scheme over the ordinary locus, which parameterizes isomorphisms

$$E[p^\infty] \xrightarrow{\sim} \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$$

Indeed: an argument identical to the one given in this section for the supersingular Igusa variety but starting with the point

$$x = [0 : 1] \in \mathbb{P}^1(\mathbb{Q}_p)$$

shows that the completed Hecke algebra of this space of p-adic automorphic forms is the same as the one appearing in Theorem 3.6.1.

Moreover, this space of p-adic automorphic forms admits an action of a very large unipotent group at p, and using this action one can produce a $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant projection operator (a type of Kirillov functor) to Katz p-adic modular forms which can in turn be used to deduce an isomorphism of completed Hecke algebras – we explain this statement and related results in [11] (cf. also [13]). In this way one obtains a proof of Theorem A that does not pass through singular/étale cohomology to show that level and weight families of classical modular forms give rise to the same completed Hecke algebra.

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