INTRODUCTION TO MIXED HODGE THEORY AND HODGE II (TALKS OF JUNE 27TH AND JULY 5TH)

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1. INTRODUCTION

We begin by paraphrasing some of the results of (pure) Hodge theory:

Definition 1. A (pure) ($\mathbb{Q}-$)Hodge structure of weight n is a $\mathbb{Q}$-vector space V together with a decomposition $V \otimes \mathbb{C} = \oplus_{p+q=n} V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$. A morphism of pure hodge structures is a map $V \to V'$ defined over $\mathbb{Q}$ and respecting the given decompositions of $V_{\mathbb{C}}$ and $V'_{\mathbb{C}}$. Equivalently, a Hodge structure of weight n is a $\mathbb{Q}$-vector space V together with a decreasing filtration F^* of $V_{\mathbb{C}}$ such that $V_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}}$ under the identification $F^p V_{\mathbb{C}} = \oplus_{p' \geq p} V^{p', n-p'}$, $V^{p,q} = F^p V_{\mathbb{C}} \cap F^q V_{\mathbb{C}}$, and a morphism of Hodge structures is a map of $\mathbb{Q}$-vector spaces respecting the filtration on $V_{\mathbb{C}}$.

Remark 2. The second definition is usually easier to work with if, for example, one wants to verify that a map is a map of Hodge structures.

Theorem 3 (Hodge theory)**.** *$X \to H^n(X, \mathbb{Q})$ is a contravariant functor from the category of compact Kahler manifolds to the category of pure hodge structures of weight n.*

In particular, the Hodge structure turns cohomology into a finer invariant when we restrict our attention to compact Kahler manifolds, with ramifications for the topology (e.g. even dimensional odd cohomology groups) and for maps between them (e.g. degeneration of Leray spectral sequence for a fibration of compact Kahler manifolds).

Living among Kahler manifolds, and of particular interest for us, are smooth projective algebraic varieties. The idea of mixed Hodge theory is to extend this refined invariant from smooth projective algebraic varieties to *all* algebraic varieties (over $\mathbb{C}$). The main result of Deligne's work (Hodge II & III) is the following theorem.

Theorem 4 (Deligne)**.** *$X \to H^n(X, \mathbb{Q})$ is a contravariant functor from the category of algebraic varieties over $\mathbb{C}$ to the category of mixed Hodge structures.*

Remark. The existence of a Hodge filtration for smooth proper (not necessarily projective) varieties is a consequence of the smooth projective case via a Lefschetz type argument and Chow (cf. Deligne - Degenerescence). If one is concerned only with quasi-projective varieties, then the techniques of Hodge II & III plus the compact Kahler case suffice. For the rest of this talk, we will restrict to this case.

The essential idea of the proof is that, via existence of compactifications and resolution of singularities, we can always "resolve" any quasi-projective variety via smooth projective varieties in some sense. This allows us to express the cohomology of any quasi-projective variety as a chain of extensions by parts of the cohomology of smooth projective varieties. The category of mixed Hodge structures keeps track of the Hodge data in these extensions. Hodge II shows how this can be done in the smooth (but not necessarily proper) case, and Hodge III treats the singular case.

In the next section we will try to motivate and understand the existence of a mixed Hodge structure in a simple but illuminating case (falling within the scope of Hodge II).

2. Introduction to Mixed Hodge Structures

A good reference for parts of this section is Griffiths and Schmid - Recent Developments in Hodge Theory.

Goal: understand Mixed Hodge Structures in the case $U = X\backslash Y$ where X is smooth projective and Y is smooth projective of (complex) codimension 1 in X.

Examples:

- X is a curve of genus g, Y is $n+1$ points, $Y = X\backslash Y$. We compute the cohomology:

i	$rk(H^i(X))$	$rk(H^i(U))$
0	1	1
1	$2g$	$2g+n$
2	1	0

  In H^1, $2g$ of the classes of U come from classes of X; the others come from the punctures. What is the "weight" of the extra n puncture classes in $H^1(U)$?
- X is $\mathbb{P}^n$, Y is a hypersurface of degree d, $U = X\backslash Y$.

How do we compute the cohomology of such a U? Can use the long exact sequence of the pair

$$... \longrightarrow H^i(X,U) \longrightarrow H^i(X) \longrightarrow H^i(U) \longrightarrow H^{i+1}(X,U) \longrightarrow ...$$

and in particular the short exact sequence

$$0 \longrightarrow H^i(X)/H^i(X,U) \longrightarrow H^i(U) \longrightarrow \ker(H^{i+1}(X,U) \to H^{i+1}(X)) \longrightarrow 0\ .$$

So, H^i has a "weight i" part arising as a quotient of $H^i(X)$ and a weight "$i+1$" part coming from a subspace of $H^{i+1}(X,U)$. For this to make sense Hodge theoretically, we should have a pure Hodge structure on $H^i(X,U)$ such that the map $H^i(X,U) \to H^i(X)$ is a map of pure Hodge structures. Given such a structure, the short exact sequence above describes $H^i(U)$ as admitting as a sub-object a natural pure Hodge structure of weight i such that the quotient has a natural pure Hodge structure of weight $i+1$. The following definition axiomatizes the kind of structure we can expect on $H^i(U)$ in order for this sequence to make sense:

Definition 5. A ($\mathbb{Q}-$)Mixed Hodge Structure is a finitely generated $\mathbb{Q}$-vector space V equipped with:

- The weight filtration: an increasing filtration (defined over $\mathbb{Q}$) $0 = W^0 \subset .... \subset W^l = V$
- The Hodge filtration: a decreasing filtration (defined over $\mathbb{C}$) $V \otimes \mathbb{C} = F^0 \supset F^1 \supset F^2 ... \supset F^m = 0$

such that for each k, $Gr^k_W(V) = W^k/W^{k-1}$ equipped with the induced Hodge filtration (by F) is a pure Hodge structure of weight k.

Example 6. In our motivating example,

$$W^k(H^i(U)) = \begin{cases} 0 & k < i \\ \operatorname{Im} H^i(X) \to H^i(U) & k = i \\ H^i(U) & k \geq i+1 \end{cases}$$

so

$$Gr^k(H^i(U)) = \begin{cases} 0 & k < i \\ \operatorname{Im} H^i(X) \to H^i(U) \cong H^i(X)/H^i(X,U) & k = i \\ H^i(U)/H^i(X) \cong \ker H^{i+1}(X,U) \to H^i(X) & k = i+1 \\ 0 & k > i+1 \end{cases}$$

But we have not yet specified the Hodge filtration on $H^i(U)$. This is a non-trivial point since the category of mixed Hodge structures is *not* semi-simple, i.e. there are non-split extensions, so simply knowing the pure Hodge structures on $Gr^k(H^i(U))$ does not pin down a Mixed Hodge structure on $H^i(U)$.

Exercise. Observe that the definition of a Hodge structure makes sense if instead of starting with a $\mathbb{Q}$-vector space we start with a finitely generated $\mathbb{Z}$-module, as do all of our construtions. Denote by $\mathbb{Z}(-n)$ the unique rank 1 pure ($\mathbb{Z}-$) hodge structure of weight $2n$ (i.e. $\mathbb{Z}(-n) = \mathbb{Z}$ as a $\mathbb{Z}$-module and $\mathbb{Z}(-n) \otimes \mathbb{C} = H^{1,1}$). Then $\operatorname{Ext}_{MHS}(\mathbb{Z}, \mathbb{Z}(-n)) = \mathbb{C}/\mathbb{Z}$.

For one specific example, however, we can already deduce the entire mixed Hodge structure on $H^i(U)$. If $X = \mathbb{P}^1$, $Y = \{0, \infty\}$ (or any two points), $U = X\backslash Y = \mathbb{G}_m$, then the short exact sequence expressing $H^1(U)$ as an extension reads:

$$0 \longrightarrow H^1(U) \longrightarrow \ker H^1(X,U) \to H^1(X) \longrightarrow 0 \ .$$

So, in this case $H^1(\mathbb{G}_m)$ has no weight 1 part, and is fact equal to the weight 2 quotient $\ker H^1(X,U) \to H^1(X)$. This is a pure Hodge structure of dimension 1 and weight 2, so it must be equal to $\mathbb{Q}(-1)$, and thus we conclude $H^1(U) = \mathbb{Q}(-1)$. In De Rham cohomology the non-trivial cohomology class can be represented by $\frac{dz}{z}$ which is equal (as a cohomology class) to $-\frac{d\bar{z}}{\bar{z}}$; thus, if we carry over the intuition from the pure case that F^p consists of classes with at least p holomorphic differentials and $\overline{F^p}$ consists of classes with at least p holomorphic differentials, we see that $\frac{dz}{z} \in F^1 \cap \overline{F^1} = "V^{(1,1)}"$.

Note that this example also shows that in order to extend Hodge theory to non-complete varieties it is necessary to allow weights different than k to appear in H^k – indeed, $H^1(\mathbb{G}_m)$ has odd dimension so it cannot possibly be pure of weight 1 (or of any odd weight). More generally, we see the necessity of allowing not just shifted but indeed mixed weights by considering X a genus $g \geq 1$ curve and removing two points from it to obtain U. Then, by any reasonable assumption of functoriality, the (non-trivial) image of $H^1(X)$ under the natural map to $H^1(U)$ should map to the "weight 1" part of $H^1(U)$, but $H^1(U)$ cannot be pure of weight 1 because it again has odd dimension (2g+1).

2.1. **The Gysin Map.** We now explain why $H^k(X,U)$ has a natural pure hodge structure such that the map $H^k(X,U) \to H^k(X)$ is a map of Hodge structures.

We first observe that we can retract U (recall $U = X\backslash Y$) to the complement V of a tubular neighborhood Y_ϵ of Y in X so that $H^i(X,U) \cong H^i(X,V)$. Then $H^i(X,V)$ is the reduced cohomology of the Thom complex $\tilde{T}_{N_X Y}$ of the normal bundle $N_X Y$ of Y in X. Since Y has complex codimension 1, the normal bundle is a rank 1 complex vector bundle over Y, and thus we obtain a Thom isomorphism

$$H^{k-2}(Y) \to \tilde{H}^k(\tilde{T}_{N_X Y}) \to H^k(X,V) \to H^k(X,U).$$

(Remark - an original choice of $i = \sqrt{-1}$ determines an orientation on the normal bundle inducing the Thom isomorphism).

Thus a natural candidate for the Hodge structure on $H^k(X,U)$ is that of $H^{k-2}(Y)$, except that this is the wrong weight if we want the composed map $H^{k-2}(Y) \to H^k(X,U) \to$

$H^k(X)$ (the Gysin map) to be a map of Pure Hodge Structures. It turns out that it is sufficient to shift the weights up by $(1,1)$, i.e. to replace $H^{k-2}(Y)$ with $H^{k-2}(Y) \otimes \mathbb{Q}(-1)$.

In other words, replacing the relative cohomology group $H^k(X, U)$ with the isomorphic $H^{k-2}(Y) \otimes \mathbb{Q}(-1)$ in the long exact sequence we obtain the Gysin sequence

$$\dots \longrightarrow H^{k-2}(Y) \otimes \mathbb{Q}(-1) \longrightarrow H^k(X) \longrightarrow H^k(U) \longrightarrow H^{k-1}(Y) \otimes \mathbb{Q}(-1) \longrightarrow \dots$$

and now we are trying to understand how this can induce a mixed Hodge structure on U. The first step is to verify that the Gysin map $H^{k-2}(Y) \otimes \mathbb{Q}(-1) \to H^k(X)$ is a map of Hodge structures. Since our concrete description of the Hodge structure on a compact Kahler manifold comes at the level of differential forms, we should attempt to understand the Gysin map at the level of differential forms.

Since X and Y are both compact, the Gysin map can be described using Poincaré duality. It is the dual of the map on the right-hand side making the following diagram commute (where X has complex dimension n so Y has complex dimension $2n-2$):

$$\begin{array}{ccccc} H^{2n-k}(X) & \xrightarrow{\cap[X]} & H_k(X) & \longrightarrow & H^k(X)^* \\ \downarrow{\iota^*} & & & & \downarrow \\ H^{2n-k}(Y) & \xrightarrow{\cap[Y]} & H_{k-2}(Y) & \longrightarrow & H^{k-2}(Y)^* \end{array}$$

If we represent cohomology classes by differential forms then the duality map $H^{2n-k}(X) \to H^k(X)^*$ is given by $\omega \mapsto (\eta \mapsto \int_X \omega \wedge \eta)$, and similarly for Y. Let us try to calculate the Gysin map in these terms. If we let $\omega \in H^{2n-k}(X)$, and denote its image in $H^k(X)^*$ by f_ω,then f_ω maps to f_{ω_Y} (=$\alpha \mapsto \int_Y \omega_Y \wedge \alpha$). To find the map $\psi : H^{k-2}(Y) \to H^k(X)$, we need to write $f_{\omega_Y}(\alpha)$ as $f_\omega(\psi(\alpha))$; i.e. we want to find a form $\psi(\alpha)$ on X such that

$$\int_Y \omega_Y \wedge \alpha = \int_X \omega \wedge \psi(\alpha).$$

- Let us consider now the example $X = \mathbb{P}^1$, $Y = \{0, \infty\}$. Then the only non-zero Gysin map will be from $H^0(Y)$ to $H^2(X)$, so in the above notation we are in the case $k = 2$. So, $\omega \in H^0(X)$ is just a constant function of value c_1 on X, and ω_Y is that constant function restricted to $Y = \{0, \infty\}$. So, for $\alpha \in H^0(X)$, $\int_Y \omega_Y \wedge \alpha = (\eta(\infty) + \eta(0)) \cdot c_1$. Thus the desired map sends α to $\alpha(\infty) + \alpha(0)$ times the volume form. In particular, it can be realized at the level of smooth forms, which is not always the case.
- Consider again the case $k = 2$, but for arbitrary smooth compact X and Y with Y consisting of a single connected component. Locally on X we can choose a uniformizer y for Y, unique up to multiplication by a non-vanishing holomorphic function ϕ. Since
$$\frac{d(\phi y)}{\phi y} = d \log \phi + \frac{dy}{y},$$
by summing over a partion of unity we can construct a form η that for any local uniformizer y is locally equal to $\frac{1}{2\pi i} \frac{dy}{y} + \theta$ where θ is a smooth (1,0)-form on X. Also fix a hermitian metric on X and let $Y_\epsilon \subset X$ be the normal ϵ-ball for sufficiently small ϵ. Let ω be a closed 2-form on X and $\alpha = c$ a constant function on Y (i.e. a representative of H^0). Then we are looking for a 2-form $\psi(c)$ on X such that
$$c \cdot \int_Y \omega_Y = \int_X \omega \wedge \psi(c)$$

Now,

$$\int_X \omega \wedge d\eta = \lim_{\epsilon \to 0} \int_{X \setminus Y_\epsilon} \omega \wedge d\eta = \lim_{\epsilon \to 0} \int_{\partial Y_\epsilon} (\omega \wedge \eta)$$

Where the second equality is from Stokes theorem (recall ω is closed so $d(\omega \wedge \eta) = \omega \wedge d\eta$). In local coordinates y,x_i with y a uniformizer for Y, then $|x|$ (size of x in the hermitian metric on the normal bundle) is equal to a non-zero multiple of $|y|$ up to first order, so for ϵ small

$$\int_{\partial Y_\epsilon} (\omega \wedge \eta) = \int (\int_{|y| \sim \epsilon} \frac{1}{2\pi i} \omega \wedge \frac{dy}{y} + \omega \wedge \theta)$$

and the limit as ϵ goes to 0 is $\int_Y \omega_Y$. (Remark - maybe there is a cleaner way to do this last part).

It is not much more complicated to see that for any α a k-form on Y and $\tilde{\alpha}$ any extension to X (which we can choose of the same type), $\psi(\alpha) = d(\eta \wedge \tilde{\alpha})$ has the desired property. Furthermore, $d(\eta \wedge \tilde{\alpha}) = d\eta \wedge \tilde{\alpha} + \eta \wedge d\tilde{\alpha}$ lies in the correct filtration ($(p+1)$-st) for this to be a map of Hodge structures.

However, there's a problem: this is not a C^∞ form on X – if $d\tilde{\alpha} \neq 0$ then it has a singularity along Y. Thus, in order to understand the Gysin map at the level of forms, we need to enlarge the class of forms we allow to include the image of this map. That is, we want to find a complex $K^* \supset A^*_X$ such that the inclusion $A^*_X \to K^*$ induces an isomorphism on cohomology and such that the map we described above can actually be defined from the cohomology of A^*_Y to the cohomology of K^*.

We also need to understand how, if we find such a complex, the analysis above remains valid as a calculation of the Gysin map. The main point to understand is that although the De Rham isomorphism $H_k \to (H^k)^*$ might not be defined via integration for a larger family of forms including η, the composed Poincaré duality isomorphism $H^{2n-k} \to (H^k)^*$ *can* potentially be defined via integration - indeed as we have just seen the integral of a form with log-singularity along Y is well-defined. We make all of this explicit:

Definition 7. Recalling that $U = X - Y$, the log complex $A^*\langle \log Y \rangle$ is the subcomplex of A^*_U generated by A^*_X and the form η above.

There is a well defined residue map R from $A^*\langle \log Y \rangle$ to $A^*_Y[1]$, which sends a form $\omega = \omega_1 \wedge \eta + \omega_2$ to $\omega_1|_Y$ (the expression of ω in this form is not unique but the map is well-defined; it is given locally by the limit of integration around smaller circles around Y).

Definition 8. The complex K^* is the kernel of R.

Note that K^* contains A^*_X and the image of the map ψ from above.

Theorem 9. *The sequence*

$$0 \longrightarrow K^* \longrightarrow A^*_X \langle \log Y \rangle \xrightarrow{R} A^*_Y[1] \longrightarrow 0$$

is exact. Furthermore,

- *The complex K^* computes the cohomology of X (isomorphism via inclusion $A^*_X \hookrightarrow K^*$),*
- *The complex $A^*_X\langle \log Y \rangle$ computes the cohomology of U (isomorphism via the inclusion $A^*_X \langle \log Y \rangle \hookrightarrow A^*_U$)*
- *The complex $A^*_Y[1]$ computes the cohomology of Y, shifted by* 1.

- *The induced long exact sequence is identified under the above isomorphisms with the Gysin sequence*

$$\dots \longrightarrow H^{k-2}(Y) \longrightarrow H^k(X) \longrightarrow H^k(U) \longrightarrow H^{k-1}(Y) \longrightarrow \dots$$

Admitting the first three bullets, we verify now that the long exact sequence is the Gysin sequence. The only point that needs checking is that the connecting homomorphism induces the Gysin map $H^{k-2}(Y) \to H^k(X)$:

Suppose α is a closed form in A_Y^{k-2} representing a class in $H^{k-2}(Y)$. Then, for any lift $\tilde{\alpha}$ to a form on X, $\tilde{\alpha} \wedge \eta \in A_X^*\langle \log Y\rangle$ has residue α. Then, $\delta[\alpha]$ is represented by $d(\tilde{\alpha} \wedge \eta)$ - note this is the map we already derived above! So, in order to see that this is the Gysin map, all that is left to check is that the definition of the Poincaré duality map $H^{2n-k}(X) \to H^k(X)^*$ in terms of smooth forms on X given by

$$\alpha \mapsto (\omega \mapsto \int_X \omega \wedge \alpha)$$

is still well defined if we allow α to be a closed form in K^*rather than A_X^*. We have already seen that these integrals are well defined, so we only need to check that they are zero if $\alpha = d\beta$ for $\beta \in K^*$. Writing $\beta = \beta_1 \wedge \eta + \beta_2$ with β_1 and β_2 smooth, for ω a closed form on X,

$$\int_X d\beta \wedge \omega = \int_X d(\beta \wedge \omega) = \lim_{\epsilon \to 0} \int_{X \setminus Y_\epsilon} d(\beta \wedge \omega) = \lim_{\epsilon \to 0} \int_{\partial Y_\epsilon} \beta \wedge \omega = \lim_{\epsilon \to 0} \int_{\partial Y_\epsilon} \pm(\omega \wedge \beta_1) \wedge \eta = \pm 2\pi i \int_Y \omega \wedge \beta_1|_Y = 0$$

the last equality since β is in the kernel of the residue map.

To verify the first two bullet points, we should view $A_X^*\langle \log Y\rangle$ as a subcomplex of A_U^* (now considering sheaves) and verify that it satisfies the Poincaré lemma, and similarly for K^* as a subcomplex of $j_* A_U^*$ containing A_X^* (where j is the inclusion $U \to X$). Details can be found in Griffiths and Schmid (Section 2).

Let's regroup. Our original goal was to use the Gysin sequence to put a mixed Hodge structure on the cohomology of U. Our first step in that direction was to ask why the Gysin map is a map of Hodge structures $H^{k-2}(Y) \to H^k(X)$, and in order to do so we had to develop a way to interpret the Gysin map at the level of differential forms. In the process of doing so, we have in fact recovered the entire Gysin sequence at the level of differential forms. Thus, we find that we have even improved upon our modest ambition: we are now in a good position not only to show that the Gysin map is a map of pure Hodge structures, but also to put a mixed Hodge structure on the cohomology of U.

Recall that the Hodge filtration on X or Y is induced by a filtration of the De Rham complex where $F^p A^*$ is generated by forms with at least p holomorphic differentials (i.e. of type (p', q), $p' \geq p$). We can take the same filtration on K^* and $A_X^*\langle \log Y\rangle$, and this filtration shifted by 1 on A_Y^* (to induce the Hodge filtration $\otimes \mathbb{Q}(-1)$ on $H^*(Y)$). The quasi-isomorphism $A_X^* \to K^*$ is strict with respect to the filtrations, and so the filtration on the cohomology of X induced by the filtration on K^* is the Hodge filtration. Furthermore, though we omit it here, it can be verified that the intrepretation of the (p,q)-th component of the cohomology of X as those cohomology classes represented by (p,q) forms remains valid even for forms in K^* (degeneration of Frohlicher spectral sequence plus K^{**} also computes $\overline{\partial}$-cohomology). We declare the filtration induced on the cohomology of U by the filtration on $A_X^*\langle \log Y\rangle$ to be the Hodge filtration.

Theorem 10. *The Gysin map* $\delta_k : H^{k-2}(Y) \otimes \mathbb{Q}(-1) \to H^k(X)$ *is a map of Hodge structures (of weight k), and thus induces Hodge structures of weight k on* $\ker \delta_k$ *and* $\operatorname{coker}\delta_k$ *. Furthermore, the Hodge filtrations on* $\ker \delta_{k+1}$ *and* $\operatorname{coker}\delta_k$ *are the same as* *those induced by the Hodge filtration on $H^k(U)$ via the short exact sequence (arising from the Gysin sequence):*

$$0 \longrightarrow \mathrm{coker}\delta_k \longrightarrow H^k(U) \longrightarrow ker\delta_{k+1} \longrightarrow 0\ .$$

Proof. We show first that the Gysin map is a map of Hodge structures: if α is a closed (p,q)-form representing a cohomology class in $H^{k-2}(Y)$, then the Gysin map takes it to $d(\tilde{\alpha}\wedge\eta)$ in $H^k(X)$, where $\tilde{\alpha}$ is an extension of α to X, which we can choose also to be a (p,q)-form.

$$d(\tilde{\alpha}\wedge\eta) = d\tilde{\alpha}\wedge\eta + \tilde{\alpha}\wedge d\eta.$$

Recall η can be expressed locally as

$$\frac{1}{2\pi i}\frac{dy}{y} + \theta$$

where θ is a smooth $(1,0)$- form. In particular, $d\eta$ is smooth, sum of $(1,1)$ and $(2,0)$, and η is of type $(1,0)$, thus $d(\tilde{\alpha}\wedge\eta) \in F^{p+1}H^k(X)$.

We now show that the quotient filtration on $coker\delta_k$ induced by $H^k(X)$ is the same as the sub-object filtration induced by $H^k(U)$. This is a natural instance of a more general framework, if we have a map $f : A \to B$ of filtered modules, then we obtain an isomorphism of modules $\mathrm{coker} f \to \mathrm{im} f$ and it is natural to ask when the induced filtrations are the same. This is the case if and only if f is strict, i.e. $\mathrm{im} f \cap F^pB = f(F^pA)$. Thus, to show these filtrations are the same, it suffices to show the map $H^k(X) \to H^k(U)$ is strict. Similarly to show that the quotient filtrations on $\ker\delta_{k+1}$ is the same as the sub-object filtration induced by $H^{k-2}(Y)\otimes\mathbb{Q}(-1)$, it suffices to show the map $H^k(X) \to H^{k-2}(Y)$ is strict.

For the first, suppose $\omega \in K^n$ is closed and its cohomology class maps to $F^pH^n(U)$. Then there is an $\alpha \in A_X^{n-1}\langle\log Y\rangle$ such that $\omega + d\alpha$ is in $F^pA_X^n\langle\log Y\rangle$. Writing $\alpha = \alpha_1\wedge\eta + \alpha_2$, $d\alpha_1|_Y \in F^pA_Y^n$ (it is the image of $(\omega + d\alpha)$ in $F^pA_Y^n$), and since Y Kahler implies (via Hodge) that the differential is strict (cf. the next section where we will show this using more homological algebra), there exists an $\alpha_1' \in F^pA_Y^n$ such that $d\alpha_1' = d\alpha_1|_Y$. Lifting α_1' to $\tilde{\alpha_1} \in F^pA_X^n$, $(\omega + d\alpha) - d(\tilde{\alpha_1}\wedge\eta)$ is an element of F^pK^n having the same image as ω in $H^n(U)$. The other proof is similar; we recast both in more general terms in the next section. □

Corollary 11. *$H^n(U)$ admits a natural mixed Hodge structure with weight filtration*

$$W^kH^n(U) = \begin{cases} 0 & k < n \\ \mathrm{im}H^n(X) & k = n \\ H^n(U) & k > n \end{cases}$$

and Hodge filtration $F^pH^n(U)$ given by classes represented by $\geq$ p-holomorphic logarithmic differential forms such that

$$Gr^kH^n(U) = \begin{cases} 0 & k < n,\ k > n+1 \\ \mathrm{coker}\delta_n & k = n \\ \ker\delta_{n+1} & k = n+1 \end{cases}$$

where the kernel and cokernels of δ_k are given their natural Hodge structures (δ_k is a map of Hodge structures $H^{k-2}(Y)\otimes\mathbb{Q}(-1) \to H^k(X)$).

The essential points in the proof were that the maps in the long exact sequence were strictly compatible with the filtrations (i.e. the natural filtrations on the image and cokernels of maps agree), and that the Gysin map was a map of Hodge structures (in fact, the strict compatibility for the Gysin map follows from the fact that it is a map of Hodge structures – any map of Hodge structures, which by definition is only required to be compatible with the filtration, is actually strictly compatible; this is the reason Hodge structures form an abelian category). In general, a short exact sequence of filtered complexes (where the filtrations on the first and last terms are the induced filtrations from the middle term) will induce a long exact sequence where the maps are compatible with the induced filtrations, however, they will not in general be strictly compatible. In the next section, we will see that if the differentials on the first and third term are strictly compatible with the filtration, then the two degree preserving maps in the long exact sequence are strictly compatible. This is a fairly rigid condition (consider, e.g., the De Rham complex with the Hodge filtration for $\mathbb{C}$ for a complex where the differentials are not strictly compatible), but as a consequence of classical Hodge theory, it is verified for compact Kahler manifolds, as we needed above.

3. The spectral sequence of a filtered complex

Let A^* be a complex equipped with a decreasing filtration F^*A^*. We will always assume that the filtration is finite on each term, i.e. for small enough p, $F^pA^n = A^n$ and for large enough p, $F^pA^n = 0$. The filtration on A^* induces a filtration (by the image) on the cohomology groups $H^k(A^*)$ and spectral sequences give a way of understanding the filtration on the cohomology and its graded components $Gr^pH^q(A^*)$.

The essential idea is that the cohomology groups can be approximated using the filtration. For example, instead of looking only at the kernel of the differential (the cycles), we can examine the set of elements moved down r levels in the filtration. As r increases, this becomes smaller, and if our filtration is bounded then for r large enough this is the same as the cycles. Similarly, instead of looking only at the image of the differential (the boundaries), we can examine the elements coming from at most r levels higher in the filtration, and if our filtration is bounded then for r large enough this is the same as the boundaries. If we performing both of these approximations simultaneously in a coherent way, then we obtain a sequence of approximations for the graded pieces of the cohomology that have the suprising advantage that they can be obtained by repeatedly taking cohomology groups of the approximation with respect to certain auxillary differentials. Such a sequence of approximations is called a spectral sequence.

Definition 12. A spectral sequence is a sequence $(E_r^{p,q})_{r\geq 0}$ together with differentials d_r such that E_{r+1} is the cohomology of E_r with respect to d_r. If for each (p,q) there is an r_0 s.t. for all $r \geq r_0$, $E_r^{p,q} = E_{r_0}^{p,q}$, then we denote $E_\infty^{p,q} = E_r^{p,q}$ for any such r and say that E_r abuts to E_∞.

Theorem 13. *Let A^* be a complex equipped with a decreasing filtration F^*A^*. Then,*

$$E_r^{p,q} := \frac{d^{-1}(F^{p+r}A^{q+1}) \cap F^pA^q}{d(F^{p+1-r}A^{q-1}) \cap F^pA^q + F^{p+1}A^q}$$

defines a spectral sequence where the differentials $d_r : E_r^{p,q} \to E_r^{p+r,q+1}$ are induced by the differential on A^*. If F^* is bounded then the sequence abuts to $E_\infty^{p,q} = Gr^pH^q(A^*)$. (Note $E_0^{p,q} = Gr^pA^q$ so $E_1^{p,q}$ is just the qth cohomology of the pth graded component of the complex).

Remark. This is not the same indexing used by Deligne (his q would be my $q-p$) which is probably more standard, but I personally find it distasteful and difficult to use, except in the applications that motivate it.

The proof is just a short computation to see that the terms are well defined and that the cohomology of one page gives the next. This spectral sequence formalizes the idea given at the start of the section.

Example 14. Consider the De Rham complex A^* (with $\mathbb{C}$ coefficients) of a complex manifold X. We can write A^n as $\oplus_{s+t=n} A^{s,t}$ as the sum of (s,t) forms, and consider the filtration $F^p A^n = \oplus_{s \geq p} A^{s,n-s}$. Then, via projection onto the (p,q)-th component have

$$E_1^{p,q} = \frac{d^{-1}(F^{p+1}A^{q+1}) \cap F^p A^q}{d(F^p A^{q-1}) \cap F^p A^q + F^{p+1} A^q} \cong \frac{\ker \overline{\partial} \cap A^{p,q}}{\overline{\partial} A^{p,q-1}} = H^{p,q}_{Dolbeault}(X) \cong H^q(\Omega^p).$$

so that $E_1^{p,q} \cong H^{p,q}_{Dolbeault}(X) \cong H^q(\Omega^p)$ and the sequence converges to $E_\infty^{p,q} = Gr^p H^q_{DR}(X)$. In particular, if X is compact Kahler then by the Hodge decomposition and dimensional considerations we deduce that this spectral sequence degenerates at the first page, i.e. $E_1 = E_\infty$. The induced filtration on cohomology is the Hodge filtration.

In general, for any double complex $A^{s,t}$ (where the differentials have degree $(1,0)$ and $(0,1)$), we can form the total complex $A^n = \oplus_n A^{s,t}$, and the same process produces a spectral sequence $E_1^{p,q} \cong H^q(A^{p,*}) \Rightarrow E_\infty^{p,q} = Gr^p H^q(A^*)$.

Lemma 15 (cf. Hodge II - 1.3.2)**.** *The following are equivalent:*

- *The spectral sequence of a filtered complex degenerates at* E_1
- $d_1 = 0$
- *The differential is strict on* A^* *(i.e.* $dF^pA^* = dA \cap F^pA^*$*).*

Proof. We first show that if the differential is strict then d_r is zero for all $r \geq 1$. Suppose $x \in d^{-1}(F^{p+r}A^{q+1}) \cap F^p A^q$ (so x represents a class in $E_r^{p,q}$). Then there exists a $y \in F^{p+r}A^q$ such that $dy = dx$ (by strictness). Since $y = 0$ in $E_r^{p,q}$ ($r \geq 1$), $dx = dy = 0$.

Clearly if d_r is zero for all $r \geq 1$ then d_1 is zero, so we now assume that $d_1 = 0$ and show d is strict. Suppose d is not strict, and let n be the largest number such that d is not strict when restricted to $F^n A^*$. Let $x \in F^n A^q$ be such that $dx \in F^{n+p}A^{q+1}$. Then, applying that the differential is zero on $E_1^{n,q}$, we have $dx \in d(F^{n+1}qA^q) + F^{n+2}A^{q+1}$. So, there is a $y \in F^{n+1}A^q$ such that $d(x-y) \in F^{n+2}A^{q+1}$, and since d is strict on F^{n+1}, there is a $z \in F^{n+2}A^q$ such that $dz = d(x-y)$. Then

$$d(y+z) = d(y+x-y) = dx$$

and $y + z \in F^{n+1}A^q$. But since d is strict on F^{n+1}, there is an $x' \in F^{n+p}A^q$ such that $dx' = dx$, and we reach a contradiction to our assumption that d was not strict on F^n. □

Corollary 16. *Let* A^* *be the De Rham complex for a compact Kahler manifold (or the generalization used in the last section for* X*). Then the differentials in* A^* *are strict with respect to the Hodge filtration.*

Proof. It suffices to verify that the spectral sequences degenerate. This follows from Example 14 in the case of the regular De Rham complex, and from the same argument once it is verified that the not only does the generalized complex compute the De Rham cohomology, but also (by looking at p,q-forms), Dolbeault cohomology. C.f. Griffiths and Schmid. □

Lemma 17 (Cf. Hodge II – 1.3.16, Lemme de deux filtrations)**.** *Let*

$$0 \longrightarrow A^* \xrightarrow{f_1} B^* \xrightarrow{f_2} C^* \longrightarrow 0$$

be an exact sequence of filtered complexes with the differentials on A^* and C^* strict and with f_1 and f_2 strict (i.e. the filtrations on A^* and C^* are induced by the filtration on B^*). Then the morphisms in the long exact sequence

$$\ldots \longrightarrow H^i(A^*) \longrightarrow H^i(B^*) \longrightarrow H^i(C^*) \longrightarrow \ldots$$

induced by f_1 and f_2 are strictly compatible with the induced filtrations, and the connecting map δ is compatible with the induced filtrations.

Proof. Let $c \in F^pZ^n(C^*)$. Lift to $\tilde{c} \in F^pB^n$ (by strictness of f_2). Then $d\tilde{c} \in \ker f_2 \cap F^pB^{n+1} = F^pA^{n+1}$, and since $\delta([c]) = [d\tilde{c}]$ we see the connecting homomorphism δ is compatible.

That f_1 and f_2 are compatible is immediate; we now show they are strictly compatible.

Suppose $a \in Z^n(A^*)$ is such that $f_1([a]) \in F^pH^n(B^*)$. Then there is a $b \in B^{n-1}$ such that $a + db \in F^pB^n$. Then $f_2(a+db) \in F^pC^n$ and $f_2(a+db) = d(f_2(b)) \in dC^{n-1}$ and since the differential on C^* is strict, $\exists c \in F^pC^{n-1}$ with $dc = f_2(a+db)$. Lifting c to $\tilde{b} \in F^pB^{n-1}$ s.t. $f_2(\tilde{b}) = c$ (strictness of f_2), we have

$$f_2(a + db - d\tilde{b}) = f_2(a + db) - d(c) = 0,$$

thus $(a+db) - d\tilde{b} \in F^pZ^n(A^*)$ and has the same image in $H^n(B^*)$ as $[a]$, so the induced map on cohomology is strict.

Finally, suppose $b \in Z^n(B^*)$ is such that $f_2([b]) \in F^pH^n(C^*)$. Then there is a $c \in C^{n-1}$ such that $f_2(b) + dc \in F^pC^n$. Let $\tilde{b} \in B^{n-1}$ be a lift of c, and let $b' \in F^pB^n$ be s.t. $f_2(b') = f_2(b) + dc$. Then

$$f_2(b' - d\tilde{b}) = f_2(b) + dc - d(f_2(\tilde{b})) = f_2(b)$$

Thus $(b' - d\tilde{b}) - b \in A^n$, and has differential db'. Thus $db' \in dA^n \cap F^pA^n$, and by strictness of the differential, $\exists a \in F^pA^n$ such that $da = db'$. Then $b' - a \in F^pZ^n(B^*)$ and $f_2(b'-a) = f_2(b') = f_2(b) + df_2(\tilde{b})$ so in cohomology they have the same image and thus the induced map on cohomology is strict. □

4. Examples

Theorem 18. *Let U be the complement of a smooth curve in $\mathbb{P}^2$. Then, for any map (of alg. varieties) $f : U \to X$ where X is projective, $f^*|_{H^i(X)} = 0$ for all $i \geq 1$.*

Proof. We examine two parts of the Gysin sequence

$$0 \longrightarrow H^1(U) \longrightarrow H^0(Y) \otimes \mathbb{Q}(-1) \longrightarrow H^2(X) \longrightarrow H^2(U) \longrightarrow H^1(Y) \otimes \mathbb{Q}(-1) \longrightarrow 0$$

and

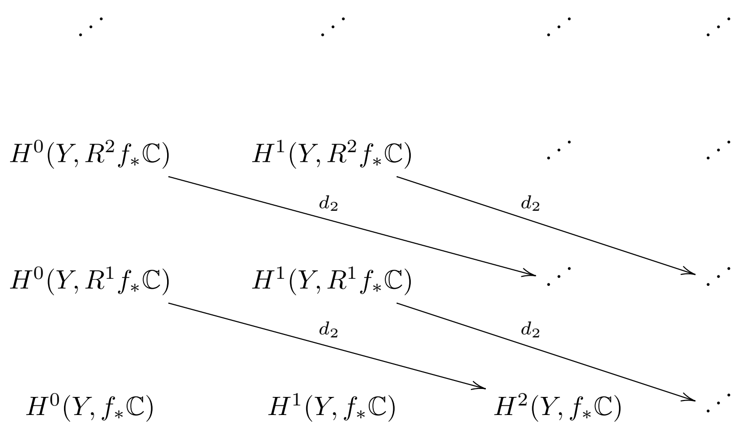
$$0 \longrightarrow H^3(U) \longrightarrow H^2(Y) \otimes \mathbb{Q}(-1) \longrightarrow H^4(X) \longrightarrow 0$$

From the first, and the fact that the Gysin map from $H^0(Y)$ is non-zero ($d\eta$ counts the number of intersections with Y, which itself represents a non-zero homology class), we deduce that $H^1(U) = 0$, and $H^2(U)$ is entirely of weight 3. The second sequence shows $H^3(U) = 0$. Thus, the only non-zero cohomology groups of U are H^0 and H^2, and H^2 is entirely of weight 3. In particular, since if X is projective $H^2(X)$ is entirely of weight ≤ 2 (we have seen this for X smooth projective where it is all of weight 2, but it is in fact true for all X projective), $f^*|_{H^2(X)}$ must be 0 (it is strictly compatible with the weight filtration). □

Example 19. Last time Dan stated the following result:

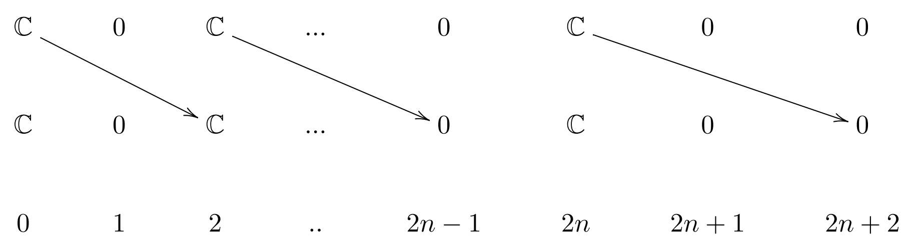
Theorem. *If $f : X \to Y$ is a submersive analytic map of compact Kahler manifolds, then the Leray spectral sequence for f degenerates at E_2 (in particular, if Y is simply connected then $H^*(X) \cong H^*(Y) \otimes H^*(F)$ for a fiber F).*

Let us write down part of the second page of this spectral sequence



The proof Dan gave analyzed the behavior of this spectral sequence with respect to the Lefschetz decomposition. Here is another way to think about it using Hodge theory: the bottom row is just the cohomology of Y, and $E_2^{p,0}$ has a Hodge structure of weight p. In general, $R^q f_*$ comes (in some sense) from the cohomology of fibers of f, so since the fibers are compact Kahler manifolds giving this cohomology a Hodge structure of weight q, $H^p(Y, R^q f_*\mathbb{C})$ "looks like" the pth cohomology group on a compact Kahler manifold with coefficients in a Hodge structure of weight q, so it is natural to guess that it should admit a Hodge structure of weight $p + q$. Indeed, if Y is simply connected then $H^p(Y, R^q f_*\mathbb{C}) = H^p(Y, \mathbb{C}) \otimes H^q(F, \mathbb{C})$ for a fiber F. Thus, we might take a leap and guess $E_r^{p,q}$ admits a Hodge structure of weight $p + q$ for all $r \geq 2$, and if we guess also that the differentials should be maps of Hodge structures, then we see immediately the degeneration: d_r maps $E_r^{p,q}$ to $E^{p+r,q-r+1}$, the first of weight $p + q$ and the second of weight $p + q + 1$, and so d_r must be 0.

Of course, we have made none of this rigorous, but nevertheless it is a useful way of looking at things. This perspective + mixed Hodge theory can also help us understand a spectral sequence that doesn't degenerate: consider the Leray spectral sequence for the standard projection $\mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$. It is not hard to check that $\mathbb{A}^{n+1}\backslash\{0\}$ has only a weight 0 one dimensional H^0 and a weight $2n + 2$ one dimensional H^1. The fibers of the map are $\mathbb{A}^1\backslash\{0\}$. Since $\mathbb{P}^n$ is simply connected, the 2nd page of the Leray spectral sequence looks like



and all of the differentials that can be non-trivial must be. This, however, is compatible with the weights, since in the top row the weight in the pth column is $p+2$ (the 2 coming from the weight 2 class in $H^1(\mathbb{A}^1\backslash\{0\})$) whereas in the bottom row the weight in the pth column is p.

Finally, we note that a more general statement of degeneration is true: if $f: X \to S$ is a smooth proper map of algebraic varieties and S is itself smooth, then the Leray spectral sequence degenerates (c.f. Hodge II - 4.1.1). This can be "seen" in a straightforward manner via weight considerations like the ones above when S is simply connected: in this case it suffices to verify that the differentials from the first column are 0 (via compatibility of the differentials with cup product in the spectral sequence), and then S smooth implies only weight $\geq p+q$ are appearing in $E^{p,q}$ whereas for $p=0$, only weight q is appearing in $E^{0,q}$ (which is the cohomology of the smooth proper fiber).

5. Hodge II

We now describe the contents of Hodge II in relation to our special case. For $U = X\backslash Y$ with X and Y both smooth projective, our strategy was as follows:

(1) We wrote down an exact sequence

$$0 \longrightarrow K^* \longrightarrow A^*_X\langle\log Y\rangle \xrightarrow{R} A^*_Y[1] \longrightarrow 0$$

of filtered complexes with the left term computing the cohomology (with Hodge structure) of X, the right term the cohomology (with Hodge structure) of Y, and the middle term the cohomology of U.

(2) We used the corresponding long exact sequence of cohomology and the rigidity of the Hodge structures on the cohomology of X and Y to show that the the filtration on the middle term induced Hodge structures on the left and right terms of the short exact sequence (coming from the long exact sequence)

$$0 \longrightarrow \mathrm{coker}\delta_k \longrightarrow H^k(U) \longrightarrow ker\delta_{k+1} \longrightarrow 0\ ,$$

thus giving $H^k(U)$ a mixed Hodge structure. The key point was that, for example, the filtration induced by $H^k(U)$ on $coker\delta_k$ (i.e. by $A^*_X\langle\log Y\rangle$) was the same as the filtration induced by $H^k(X)$ (i.e. by K^*).

We cannot, however, expect a general smooth variety U to admit an embedding of this form. Instead, we must allow Y to be a union of smooth projective hypersurfaces with transverse intersections (a normal crossings divisor). There is a natural generalization of $A^*_X\langle\log Y\rangle$ to this case calculating the cohomology of U, however, we can no longer write this complex as an extension of two complexes associated to smooth projective varieties. Instead, it has a natural increasing filtration $W^*A^*_X\langle\log Y\rangle$ (in the case above $W^0 = K^*$, $W^1 = A^*_X\langle\log Y\rangle$ so the filtration is the same as the short exact sequence) such that the graded components compute the cohomology of smooth projective varieties (given by intersections of the hypersurfaces in Y). Then, rather than a long exact sequence computing the cohomology of U out of smooth projective varieties, we have a spectral sequence (corresponding to the filtration W^*) with E_1 page the cohomology of smooth projective varieties that computes the cohomology of U. The spectral sequence expresses the E_∞ page, i.e. the graded components of $H^*(U)$ as sub-quotients of the E_1 page, and in fact due to compatibility of the differentials, this induces a Hodge structure on the graded components of $H^*(U)$ (via the Hodge structures on the E_1 page). Verifying that these Hodge structures on the graded components actually come from the filtration induced by our original filtration on $H^*(U)$ is analogous to verifying it in the case of the long exact sequence (in fact, the long exact sequence arises naturally from the spectral sequence associated to the two-term filtration above).

Thus, Section 1 of Hodge II is concerned largely with the exposition of the homological algebra necessary to track filtrations through spectral sequences. The most important result is 1.3.16 - The Two Filtrations Lemma, which generalizes Lemma 17 from above and is used to show that, just like with the long exact sequence, the (Hodge) filtration on $H^*(U)$ will agree on graded components with the Hodge filtration induced by the smooth projective varieties appearing in the weight spectral sequence (so that the "Hodge" filtration on $H^*(U)$ earns its name, i.e. it induces a mixed Hodge structure).

Section 2 of Hodge II is an exposition of the basic properties of mixed Hodge structures, viewed independent of their role in complex geometry. The most important result here is 2.3.5, which says that mixed Hodge structures form an abelian category. Note this is not at all obvious, as, for example, filtered vector spaces do not form an abelian category. The difficultly is that for a filtered morphism in general there is no reason for the cokernel with its quotient topology to have the same filtration as the image with its sub-object topology – indeed we have already seen that this is the case only when the morphism is strict. Thus, the fact that MHS is an abelian category is intimately tied to the fact that morphisms in MHS are automatically strict with respect to all filtrations, and we have already seen how this strictness/abelianness comes into play when we defined the Hodge structure in our example with the long exact sequence.

Section 3 is the heart of the paper; it develops the basic properties of the log complex and then uses the results of section 1 (just as we have done here) to produce a mixed Hodge structure from a normal crossings compactification. The main element is the construction; that this mixed Hodge structure is unique and functorial follows from the general setup for the computation and basic properties of how we can choose the compactification. The end of section 3 explicits some further properties of the mixed Hodge structure appearing (e.g. bounds on Hodge numbers) that result from the construction.

Finally, Section 4 discusses some applications; for example, the semi-simplicity theorem, a precursor to the semi-simplicity part of the decomposition theorem (c.f. De Cataldo for a discussion of this).